Araki-Wyss representation

CLAUDE-ALAIN PILLET

CPT-CNRS, UMR 6207
Université du Sud Toulon-Var
B.P. 20132
83957 La Garde Cedex, France
pillet@univ-tln.fr

1 Non-interacting fermions

A system of non-interacting fermions is specified by a one particle Hilbert space $\mathfrak{h}$ and a one-particle Hamiltonian h, a self-adjoint operator on $\mathfrak{h}$. Within the C^*-algebraic approach, observables of this system are element of the C^*-algebra of Canonical Anticommutation Relations $\mathrm{CAR}(\mathfrak{h})$. Their time evolution is given by the group of Bogoliubov automorphisms

$$\tau^t(a(f)) = a(\mathrm{e}^{\mathrm{i}th} f),$$

associated to h. Thus, the dynamics of the system is described by the C^*-dynamical system $(\mathrm{CAR}(\mathfrak{h}), \tau)$. Taking gauge-invariance into account we should in fact restrict the algebra to its gauge-invariant part $\mathrm{CAR}_0(\mathfrak{h})$ (see Section 2 in [Fock and non-Fock states on CAR-algebras]). It is often more convenient to keep the full CAR algebra and consider only gauge-invariant states instead.

2 Gauge-invariant quasi-free states

In the Fock representation the dynamical group τ as well as the gauge group ϑ are unitarily implemented by the second quantized Hamiltonian $H = \mathrm{d}\Gamma(h)$ and the number operator $N = \mathrm{d}\Gamma(I)$,

$$\pi_F(\tau^t(A)) = \mathrm{e}^{\mathrm{i}tH}\pi_F(A)\mathrm{e}^{-\mathrm{i}tH}, \pi_F(\vartheta^\varphi(A)) = \mathrm{e}^{\mathrm{i}\varphi N}\pi_F(A)\mathrm{e}^{-\mathrm{i}\varphi N}.$$

The Hamiltonian h of a single fermion confined in a finite volume $\Lambda \subset \mathbb{R}^d$ typically has purely discrete spectrum and $\mathrm{e}^{-\beta h}$ is trace class for any $\beta > 0$. Using the identity

$$\det(I + A) = \mathrm{tr}(\Gamma(A)), \tag{1}$$

we conclude that $\mathrm{tr}(\mathrm{e}^{-\beta(H-\mu N)}) = \det(I + \mathrm{e}^{-\beta(h-\mu)})$ for any $\beta > 0$ and $\mu \in \mathbb{R}$. Hence $\mathrm{e}^{-\beta(H-\mu N)}$ is also trace class and the Gibbs grand canonical ensemble at inverse temperature β and chemical potential μ is a gauge-invariant Fock state with density matrix

$$\rho_{\beta\mu} = \frac{\mathrm{e}^{-\beta(H-\mu N)}}{\mathrm{tr}(\mathrm{e}^{-\beta(H-\mu N)})}.$$

It is the unique β-KMS state on $\mathrm{CAR}(\mathfrak{h})$ for the dynamics $t \mapsto \tau^t \circ \vartheta^{-\mu t}$. Using again identity (1) a simple calculation shows that the characteristic function of this state is given by

$$E_{\beta\mu}(u) = \det(I + (u - I)f_{\beta\mu}(h)), \tag{2}$$

where

$$f_{\beta\mu}(\varepsilon) = \frac{1}{1 + \mathrm{e}^{\beta(\varepsilon-\mu)}},$$

is the Fermi-Dirac distribution function.

Since $u - I$ is finite rank the characteristic function (2) still makes sense in the infinite volume limit, despite of the fact that the Boltzmann weight $\mathrm{e}^{-\beta(H-\mu N)}$ is no more trace class in this limit. One can show directly that (2) is the characteristic function of the unique β-KMS state for the group $t \mapsto \tau^t \circ \vartheta^{-\mu t}$. Its restriction to the gauge-invariant sub-algebra $\mathrm{CAR}_0(\mathfrak{h})$ is therefore a (τ, β)-KMS state.

More generally one has the following

Theorem 1 *Let T be a self-adjoint operator on the Hilbert space $\mathfrak{h}$. If $0 \le T \le I$ then $E(u) = \det(I + (u - I)T)$ is the characteristic function of a gauge-invariant state ω_T on $\mathrm{CAR}(\mathfrak{h})$. ω_T is called the gauge-invariant quasi-free state generated by T. Equivalent ways to characterize this state are*

1. *For all $f_1, \dots, f_n \in \mathfrak{h}$ and $g_1, \dots, g_m \in \mathfrak{h}$ one has*

$$\omega_T(a^*(f_1) \cdots a^*(f_n) a(g_m) \cdots a(g_1)) = \delta_{nm} \det\{(g_i | T f_j)\}.$$

2. *For $f \in \mathfrak{h}$ set $\varphi(f) = 2^{-1/2}(a(f) + a^*(f))$. The Wick theorem*

$$\begin{aligned} \omega_T(\varphi(f_1) \cdots \varphi(f_{2n+1})) &= 0, \\ \omega_T(\varphi(f_1) \cdots \varphi(f_{2n})) &= \sum_{\pi \in \mathcal{P}_n} \epsilon(\pi) \prod_{j=1}^{n} \omega_T(\varphi(f_{\pi(2j-1)}) \varphi(f_{\pi(2j)})), \end{aligned}$$

   holds. In the last expression, the sum runs over the set $\mathcal{P}_n$ of pairings, i.e., permutations π of $\{1, \dots, 2n\}$ such that $\pi(2j-1) < \pi(2j)$ and $\pi(2j-1) < \pi(2j+1)$. Moreover, $\epsilon(\pi)$ denotes the signature of the permutation π.

The state ω_T also has an information theoretic characterization: it has maximal entropy among all the gauge-invariant states ν such that $\nu(a^*(f)a(g)) = (g, Tf)$ in the following sense. For a finite dimensional subspace $\mathfrak{K} \subset \mathfrak{h}$ define the entropy

$$S(\nu | \mathfrak{K}) = -\mathrm{tr}(\rho \log \rho)$$

where ρ is the density matrix of the restriction of ν to the finite dimensional algebra $\mathrm{CAR}(\mathfrak{K})$. Then

$$S(\omega_T | \mathfrak{K}) = \max_{\nu \in E_T} S(\nu | \mathfrak{K}),$$

where E_T denotes the set of gauge-invariant states ν such that $\nu(a^*(f)a(g)) = (g, Tf)$. This follows from a simple adaptation of the proof of Proposition 1a in [LR].

We refer the reader to [A] and [BR2] for more information on quasi-free states on $\mathrm{CAR}(\mathfrak{h})$.

3 Araki-Wyss representation

Let ω_T be the gauge-invariant quasi-free state on $\mathrm{CAR}(\mathfrak{h})$ generated by T. The associated GNS representation, which we denote by $(\mathcal{H}_T, \pi_T, \Omega_T)$, was first constructed by Araki and Wyss in [AW]. It can be described as follows:

1. $\mathcal{H}_T = \Gamma_{\mathrm{a}}(\mathfrak{h}_1 \oplus \mathfrak{h}_2) \subset \Gamma_{\mathrm{a}}(\mathfrak{h} \oplus \mathfrak{h})$ where $\mathfrak{h}_1 = (\mathrm{Ran}(I - T))^{\mathrm{cl}}$ and $\mathfrak{h}_2 = (\mathrm{Ran} T)^{\mathrm{cl}}$.

2. $\Omega_T = \Omega_{\mathrm{F}}$, the Fock vacuum vector.

3. The $*$-morphism π_T is given by

$$\pi_T(a(f)) = a\left(\sqrt{I - T} f \oplus 0\right) + a^*\left(0 \oplus \overline{\sqrt{T} f}\right),$$

   where $\overline{\cdot}$ denotes an arbitrary complex conjugation on $\mathfrak{h}$.

Using the exponential law for fermions $U : \Gamma_{\rm a}(\mathfrak{h}_1 \oplus \mathfrak{h}_2) \to \Gamma_{\rm a}(\mathfrak{h}_1) \otimes \Gamma_{\rm a}(\mathfrak{h}_2)$ an equivalent representation with cyclic vector $U\Omega_T = \Omega_{\rm F} \otimes \Omega_{\rm F}$ is obtained. It is explicitly given by

$$U\pi_T(a(f))U^* = a\left(\sqrt{I-T}f\right) \otimes I + \Theta \otimes a^*\left(\sqrt{T}f\right),$$

where $\Theta = \Gamma(-I) = (-1)^N$. In the limiting cases $T = 0$ and $T = I$ which correspond to the vacuum state **vac** and to the filled Fermi sea **full** the Araki-Wyss representation degenerates to the Fock and anti-Fock representations π_F and π_{AF}.

The reader should consult [A] and [D] for a detailed introduction to quasi-free representations of $\mathrm{CAR}(\mathfrak{h})$.

4 Enveloping von Neumann algebra

The following theorem summarizes some interesting features of the enveloping von Neumann algebra $\mathfrak{M}_T = \pi_T(\mathrm{CAR}(\mathfrak{h}))''$ of a gauge-invariant quasi-free state ω_T.

Theorem 2 *1. ω_T is primary, i.e., its enveloping von Neumann algebra is a factor. $\mathfrak{M}_T$ is of type*

I if either $\mathfrak{h}$ is finite dimensional or $\mathfrak{h}$ is infinite dimensional and $T = 0$ or $T = I$.

II if $\mathfrak{h}$ is infinite dimensional and $T = I/2$.

III_λ if $\mathfrak{h}$ is infinite dimensional and $T = (I + \lambda^{\pm 1})^{-1}$ for some $\lambda \in]0, 1[$.

III_1 if the continuous spectrum of T is not empty.

2. ω_T is modular, i.e., the cyclic vector Ω_T is separating for $\mathfrak{M}_T$, if and only if $\mathrm{Ker}T = \mathrm{Ker}(I - T) = \{0\}$.

3. ω_T and ω_S are quasi-equivalent if and only if the operators $T^{1/2} - S^{1/2}$ and $(I - T)^{1/2} - (I - S)^{1/2}$ are Hilbert-Schmidt.

If $\mathrm{Ker}\, T = \mathrm{Ker}(I - T) = \{0\}$ then modular theory applies to $\mathfrak{M}_T$ (see [Tomita-Takesaki theory]). On $\mathcal{H}_T$ there exist an anti-unitary involution J (the modular conjugation) and a positive operator Δ (the modular operator) such that

$$J\Delta^{1/2}A\Omega_T = A^*\Omega_T,$$

for all $A \in \mathfrak{M}_T$. These operators are explicitly given by

$$J = (-1)^{N(N-1)/2}\Gamma(j), \quad \Delta = \Gamma(\mathrm{e}^s \oplus \mathrm{e}^{-\bar{s}}),$$

where $j : f \oplus g \mapsto \overline{g \oplus f}$ and $s = \log T(I - T)^{-1}$.

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