Full Counting Statistics and Time's Arrow in Open Quantum Systems

Joint work with V. Jakšić (McGill University), Y. Ogata (Tokyo University) and R. Seiringer (McGill University)

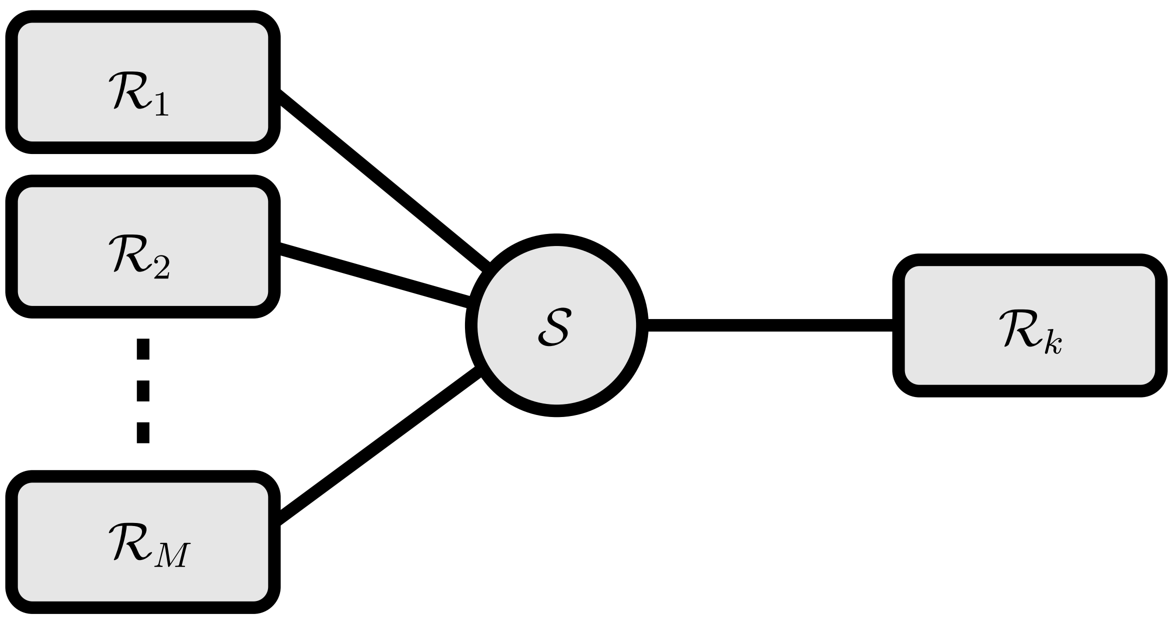
C.-A. Pillet

Centre de Physique Théorique – UMR 6207

CNRS – Aix-Marseille Université – Université du Sud Toulon-Var

ANR Ham-Mark 2011

OPEN QUANTUM SYSTEMS



HISTORICAL BACKGROUND

Pre-History [$<$ 1950]: The Atom–Radiation Problem

Phenomenological approach $+$ perturbation theory
Environment completely encoded into transition rates

- Einstein's "$A - B$" law (1917).
- Dirac calculation of A and B (1928).
- Pauli's Master equation (1928).
- Bloch equation (1946).
- Fermi's "Golden Rule" (1948).

$\Downarrow$

Markovian Dynamics

HISTORICAL BACKGROUND

The 60': A Quantum Theory of Dissipation

Microscopic dynamics of coupled system

- Nonequilibrium statistical mechanics, environment induced fluctuations, quantum Brownian motion
  - Projection methods, exact master equations:
    van Hove (1955), Nakajima (1958), Zwanzig (1960), Montroll (1960), Prigogine-Resibois (1961)
  - Lagrangian formulation, path integrals:
    Schwinger (1961), Feynman-Vernon (1963), Keldysh (1965)
  - Hamiltonian formulation, quantum Langevin equation:
    Ford-Kac-Mazur (1965)
- Effects of dissipation on quantum phenomena (interferences, tunneling, ...)
  - Instanton analysis:
    Callan-Coleman (1977), Caldeira-Leggett ($\geq$ 1981)

HISTORICAL BACKGROUND

The 70': Early Mathematical Results

- Quantum stochastic processes, dynamical semigroups
  - Davies ($\geq$ 1969)
  - Lindblad Gorini-Kossakowski-Sudarshan Accardi (1976)
- Weak and singular coupling limits
  - Hepp-Lieb (1973), Davies ($\geq$ 1974)
- Return to equilibrium and nonequilibrium steady states
  - Robinson (1973)
  - Davies (1978), Lebowitz-Spohn (1978), Davies-Spohn (1978)

$$\Downarrow$$

Return of the Markovian dynamics

Rise of the algebraic approach to QM

Why the Algebraic Approach ?

- Study (interesting) large time behavior of $\mathcal{S}$ coupled to its environment, e.g.,
  - return to equilibrium
  - relaxation to steady state
- Idealize “large time” by “$\lim\limits_{t\to\infty}$”
- (interesting) requires “continuous spectrum”
- Need to take thermodynamic limit of the environment
- TD-limit and $t \to \infty$ can not be exchanged
- Need to describe quantum dynamics of infinitely many degrees of freedom

HISTORICAL BACKGROUND

After 1990: Following the Algebraic Way

- Equilibrium properties of the spin-boson model
  - Spohn (1989), Hübner-Spohn (1995), Arai-Hirokawa (1997)
- Return to equilibrium in Pauli-Fierz models
  - Jakšić-P (1995-6), Bach-Fröhlich-Sigal (2000), Derezinski-Jakšić (2003), Fröhlich-Merkli (2004)
- Nonequilibrium steady states
  - Liouvillian approach:
    Jakšić-P (2002), Merkli-Mueck-Sigal (2007)
  - Ruelle's scattering approach, Büttiker-Landauer formalism:
    Ruelle (1999), Fröhlich-Merkli-Ueltschi (2003), Aschbacher-P (2003), Aschbacher-Jakšić-Pautrat-P (2007), Nenciu (2007)
  - Entropy production:
    Ruelle (2001), Jakšić-P (2001)
- Fluctuation-Dissipation (linear-response) theory
  - Jakšić-Ogata-P (2006), Jakšić-Pautrat-P (2009)

ENTROPY PRODUCTION

- Finite quantum system $\mathcal{Q} = (\mathcal{O}, \tau^t, \omega)$: finite dimensional Hilbert space $\mathcal{H}$, self-adjoint Hamiltonian H and density matrix ω

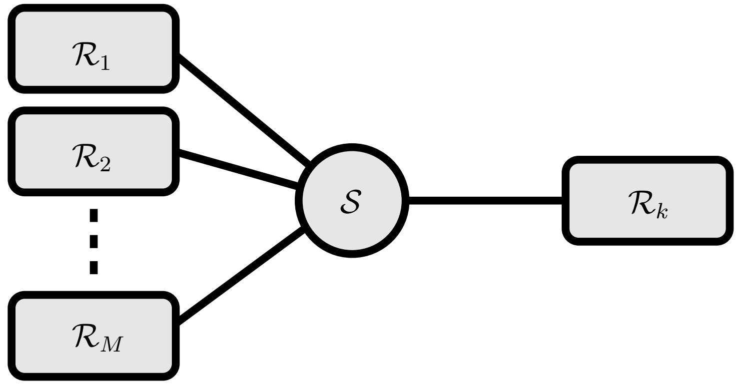
$$\mathcal{O} = \mathcal{L}(\mathcal{H}), \qquad \tau^t(A) = \mathrm{e}^{\mathrm{i}tH} A \mathrm{e}^{-\mathrm{i}tH}, \qquad \omega(A) = \mathrm{Tr}\,(\omega A)$$

- Heisenberg/Schrödinger pictures: $A_t = \tau^t(A), \qquad \omega_t = \omega \circ \tau^t, \qquad \omega(A_t) = \omega_t(A)$
- Entropy observable: $S = -\log\omega \Rightarrow S_t = \tau^t(S) = -\log\omega_{-t}$
- Relative entropy: $0 \geq S(\omega_t|\omega) = \mathrm{Tr}\,(\omega_t(\log\omega - \log\omega_t) = -\omega(S_t - S)$
- Entropy production observable ($\simeq$ *phase space contraction rate*)

$$\sigma = \left.\frac{\mathrm{d}}{\mathrm{d}t} S_t\right|_{t=0} = \mathrm{i}[H, S]$$

- Mean entropy production rate: $\Sigma^t = \dfrac{S_t - S}{t} = \dfrac{1}{t}\displaystyle\int_0^t \sigma_s\,\mathrm{d}s$
- Entropy balance: $\omega(\Sigma^t) = \dfrac{1}{t}\displaystyle\int_0^t \omega(\sigma_s)\,\mathrm{d}s = -\dfrac{1}{t} S(\omega_t|\omega) \geq 0$

OPEN SYSTEMS



- $$\mathcal{H} = \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}_1} \otimes \cdots , \qquad \mathcal{O} = \mathcal{O}_{\mathcal{S}} \otimes \mathcal{O}_{\mathcal{R}_1} \otimes \cdots , \qquad \omega = \omega_{\mathcal{S}} \otimes \omega_{\mathcal{R}_1} \cdots$$
  $$H = H_{\mathcal{S}} + H_{\mathcal{R}_1} + \cdots + V_1 + \cdots , \quad V_j \in \mathcal{O}_{\mathcal{S}} \otimes \mathcal{O}_{\mathcal{R}_j}$$
- $$\omega_{\mathcal{S}} = \mathrm{dim}\mathcal{H}_{\mathcal{S}}{}^{-1} I, \qquad \omega_{\mathcal{R}_j} = \frac{\mathrm{e}^{-\beta_j (H_{\mathcal{R}_j} - \mu_j N_j)}}{\mathrm{Tr}\,(\mathrm{e}^{-\beta_j (H_{\mathcal{R}_j} - \mu_j N_j)})}$$

with $[H_{\mathcal{R}_j}, N_j] = 0$ (conserved charges)

OPEN SYSTEMS

- Entropy observable: $S = -\log\omega = \sum_j \beta_j(H_{\mathcal{R}_j} - \mu_j N_j) + \text{const.}$
- Entropy production

$$\sigma = -\sum_j \beta_j\,(\Phi_j - \mu_j \mathcal{J}_j)$$

where

$$\Phi_j = -\mathrm{i}[H, H_{\mathcal{R}_j}], \qquad \mathcal{J}_j = -\mathrm{i}[H, N_j]$$

are the energy and charge currents out of reservoir $\mathcal{R}_j$

- Mean entropy production rate

$$\Sigma^t = -\sum_j \beta_j \left(\frac{H_{\mathcal{R}_j} - \tau^t(H_{\mathcal{R}_j})}{t} - \mu_j \frac{N_j - \tau^t(N_j)}{t} \right)$$

$$= -\frac{1}{t}\int_0^t \sum_j \beta_j(\Phi_{js} - \mu_j \mathcal{J}_{js})\mathrm{d}s$$

TIME REVERSAL INVARIANCE (TRI)

- If H and ω are real matrices w.r.t. some orthogonal basis of $\mathcal{H}$ then the system is time reversal invariant. Let $\theta : \mathcal{H} \to \mathcal{H}$ be complex conjugation in that particular basis and $\Theta(A) = \theta A \theta^{-1}$, then

$$\Theta \circ \tau^t = \tau^{-t} \circ \Theta, \qquad \omega \circ \Theta(A) = \omega(A^*)$$

- $\Theta(\sigma) = -\sigma, \qquad \tau^t \circ \Theta(\Sigma^t) = -\Sigma^t \Rightarrow$ the spectrum of Σ^t is symmetric around 0
- Let $\Sigma^t = \sum_{s \in \mathrm{sp}(\Sigma^t)} s P_s$. Unless $[H, S] = 0$, the Evans-Searles fluctuation relation

$$\frac{\omega(P_{-s})}{\omega(P_s)} = \mathrm{e}^{-ts}$$

does not hold: departure from classical mechanics, purely quantum effect

- Equivalent formulation: the cumulant generating function $e_t(\alpha) = \log \omega \left(\mathrm{e}^{-\alpha t \Sigma^t} \right)$ does not satisfy

$$e_t(1 - \alpha) = e_t(\alpha)$$

A SPECTRAL FLUCTUATION THEOREM

- Assume $\omega > 0$. The entropic functional (Rényi relative entropy)

$$e_{2,t}(\alpha) = S_\alpha(\omega_t|\omega) = \log \mathrm{Tr}\,(\omega_t^\alpha \omega^{1-\alpha})$$

has a spectral interpretation (notice: $[H,S]=0 \Rightarrow e_{2,t}(\alpha) = e_t(\alpha)$)

- On the Hilbert space $\mathcal{O}$ equipped with the Hilbert-Schmidt inner product $(A|B) = \mathrm{Tr}\,(A^*B)$ the operator

$$\Delta_{\omega_t|\omega} : A \mapsto \omega_t A \omega^{-1}$$

is positive and one has

$$e_{2,t}(\alpha) = \log(\Omega_\omega|\Delta_{\omega_t|\omega}^\alpha \Omega_\omega)$$

where $\Omega_\omega = \omega^{1/2}$.

- Let Q^t be the spectral measure of $-t^{-1}\log\Delta_{\omega_t|\omega}$ for Ω_ω. Then

$$e_{2,t}(\alpha) = \log \int \mathrm{e}^{-\alpha t x}\mathrm{d}Q^t(x)$$

A SPECTRAL FLUCTUATION THEOREM

- If the system is TRI, then

$$e_{2,t}(1-\alpha) = \log \mathrm{Tr}\,(\omega_t^{1-\alpha}\omega^{\alpha}) = \log \mathrm{Tr}\,(\omega^{1-\alpha}\omega_{-t}^{\alpha}) = \log \mathrm{Tr}\,(\omega^{1-\alpha}\omega_t^{\alpha}) = e_{2,t}(\alpha)$$

- The Evans-Searles symmetry $e_{2,t}(1-\alpha) = e_{2,t}(\alpha)$ yields

$$\int \mathrm{e}^{-\alpha t x}\mathrm{d}Q^t(x) = \int \mathrm{e}^{\alpha t x}\mathrm{e}^{-tx}\mathrm{d}Q^t(x) = \int \mathrm{e}^{-\alpha t x}\mathrm{e}^{-tx}\mathrm{d}Q^t(-x)$$

- Thus, Evans-Searles symmetry translates into [Matsui-Tasaki 2003]

$$\frac{\mathrm{d}Q^t(-x)}{\mathrm{d}Q^t(x)} = \mathrm{e}^{-tx}$$

  i.e., interpreting Q^t as the law of a random variable X_t, we have a fluctuation theorem for X_t: negative values of X_t are exponentially less probable than positive ones!

- Problem: $-t^{-1}\log\Delta_{\omega_t|\omega}$ is not observable! What is the physical interpretation of X_t ?

FULL COUNTING STATISTICS

- At time t = 0, with the system in the state ω, one performs a measurement of the entropy observable $S = \log \omega$. Possible outcomes are eigenvalues of S and s is observed with probability $\omega(P_s)$. After the measurement, the state of the system reduces to

$$\frac{\omega P_s}{\omega(P_s)}$$

and this state evolves over the time interval $[0, t]$ to

$$\frac{\mathrm{e}^{-\mathrm{i}tH}\omega P_s \mathrm{e}^{\mathrm{i}tH}}{\omega(P_s)}$$

A second measurement of S at time t yields the result s' with probability

$$\frac{\mathrm{Tr}\ \left(\mathrm{e}^{-\mathrm{i}tH}\omega P_s \mathrm{e}^{\mathrm{i}tH} P_{s'}\right)}{\omega(P_s)}$$

FULL COUNTING STATISTICS

- The joint probability distribution of the two measurement is given by

$$\mathrm{Tr}\left(\mathrm{e}^{-\mathrm{i}tH}\omega P_s \mathrm{e}^{\mathrm{i}tH} P_{s'}\right)$$

  and the probability distribution of the mean rate of change of entropy, $\phi = (s' - s)/t$, is given by

$$\mathbb{P}^t(\phi) = \sum_{s'-s=t\phi} \mathrm{Tr}\left(\mathrm{e}^{-\mathrm{i}tH}\omega P_s \mathrm{e}^{\mathrm{i}tH} P_{s'}\right)$$

- $\mathbb{P}^t$ is the full counting statistics of S for the time interval $[0, t]$ (Levitov-Lesovik 1993, Kurchan 2000).

FULL COUNTING STATISTICS

- Link:

$$\sum_{\phi} \mathbb{P}_t(\phi) \mathrm{e}^{-\alpha t \phi} = \mathrm{Tr}\, \omega_t^{1-\alpha} \omega^{\alpha}$$

  TRI implies

$$\mathrm{Tr}\, \omega_t^{1-\alpha} \omega^{\alpha} = \mathrm{Tr}\, \omega_t^{\alpha} \omega^{1-\alpha}$$

  hence

$$\mathrm{d}Q^t(x) = \sum_{\phi} \mathbb{P}_t(\phi) \delta(x - \phi) \mathrm{d}x$$

  i.e., the random variable X_t describes the entropy change between the two measurements of S.

- Fluctuation theorem

$$\frac{\mathbb{P}_t(-\phi)}{\mathbb{P}_t(\phi)} = \mathrm{e}^{-t\phi}$$

- Remark: a similar result holds for the FCS of individual energy/charge transfer

THERMODYNAMIC LIMIT

Sequence $\mathcal{Q}_M = (\mathcal{O}_M, \tau_M^t, \omega_M)$ of finite quantum systems with associated objects

$$\sigma_M, \qquad \varsigma_{\omega_M}^t(\,\cdot\,) = \omega_M^{\mathrm{i}t} \cdot \omega_M^{-\mathrm{i}t}$$

Assuming: $\mathcal{Q}_M \to \mathcal{Q} = (\mathcal{O}, \tau^t, \omega)$, i.e.,

- there is a C^*-algebra $\mathcal{O}$ in which the $\mathcal{O}_M$ can be consistently embedded

$$\mathcal{O}_M \subset \mathcal{O}_{M+1} \subset \mathcal{O}$$

so that

$$\mathcal{O}_{\mathrm{loc}} = \bigcup_M \mathcal{O}_M \subset \mathcal{O}$$

densely.

- for all $A \in \mathcal{O}_{\mathrm{loc}}$

$$\lim_{M\to\infty} \omega_M(A) = \omega(A), \quad \lim_{M\to\infty} \tau_M^t(A) = \tau^t(A), \quad \lim_{M\to\infty} \varsigma_{\omega_M}^t(A) = \varsigma_\omega^t(A)$$

exists uniformly on compact time intervals

THERMODYNAMIC LIMIT

- $$\lim_{M\to\infty} \sigma_M = \sigma$$

  exists in the norm of $\mathcal{O}$.

- $$e_{2,t}(\alpha) = \lim_{M\to\infty} e_{2,t,M}(\alpha)$$

  exists and is finite for $\alpha, t \in \mathbb{R}$

one can conclude that

- $\mathbb{P}_{t,M}$ converges weakly to a Borel probability measure $\mathbb{P}_t$ such that

  $$e_{2,t}(\alpha) = \log \int \mathrm{e}^{-\alpha t \phi} \mathrm{d}\mathbb{P}_t(\phi)$$

THERMODYNAMIC LIMIT

- $e_{2,t}(\alpha)$ is a real analytic convex function of $\alpha \in \mathbb{R}$
- All cumulants of $\mathbb{P}_{t,M}$ converge to the corresponding cumulants of $\mathbb{P}_t$
- TRI $\Rightarrow$ fluctuation theorem

$$\frac{\mathrm{d}\mathbb{P}_t(-\phi)}{\mathrm{d}\mathbb{P}_t(\phi)} = \mathrm{e}^{-t\phi}$$

- The entire modular structure survives thermodynamic limit. It now lives in a standard representation of the von Neumann algebra $\mathfrak{M} = \pi_\omega(\mathcal{O})''$, where π_ω is the GNS representation of $\mathcal{O}$ induced by the state ω. In particular, the relative modular operator $\Delta_{\omega_t|\omega}$ becomes a positive operator on the GNS Hilbert space.
- $\mathbb{P}_t$ is the spectral measure for $-\frac{1}{t}\Delta_{\omega_t|\omega}$ and Ω_ω
- $\mathbb{P}_t$ does not refer to a “real” measurement process, but merely to an idealized one.

THERMODYNAMIC LIMIT

- Entropy observable S does not survive the thermodynamic limit, but $S_t - S$ becomes Araki's relative hamiltonian $\ell_{\omega_{-t}|\omega} \in \mathfrak{M}$ and

$$\Sigma^t = \frac{1}{t}\ell_{\omega_{-t}|\omega} = \frac{1}{t}\int_0^t \sigma_s \mathrm{d}s$$

- The entropy production observable σ generates the entropy cocycle $c^t = \tau^t(\ell_{\omega_t|\omega})$ $(c^{t+s} = c^t + \tau^t(c^s))$

$$\sigma = \left.\frac{\mathrm{d}}{\mathrm{d}t}c^t\right|_{t=0}$$

- Σ^t is related to Araki's relative entropy by

$$\omega(\Sigma^t) = -\frac{1}{t}S(\omega_t|\omega) = -\frac{1}{t}(\Omega_{\omega_t}|\log \Delta_{\omega|\omega_t}\Omega_{\omega_t}) \geq 0$$

- $$\omega(\Sigma^t) = \mathbb{E}^t(\phi), \qquad \omega((\Sigma^t)^2) - \omega(\Sigma^t)^2 = \mathbb{E}^t(\phi^2) - \mathbb{E}^t(\phi)^2$$

  but this fails for higher order cumulants

LARGE TIME LIMIT

Assume:

$$e_2(\alpha) = \lim_{t\to\infty} \frac{1}{t} e_{2,t}(\alpha) = \lim_{t\to\infty} \frac{1}{t} \log \int \mathrm{e}^{-\alpha t \phi} \mathrm{d}\mathbb{P}_t(\phi)$$

exists, is finite and differentiable on $\mathbb{I} =]-\delta, 1+\delta[$ for some $\delta > 0$ and

$$\langle \sigma \rangle_+ = -e_2'(0) > 0$$

Then one has:

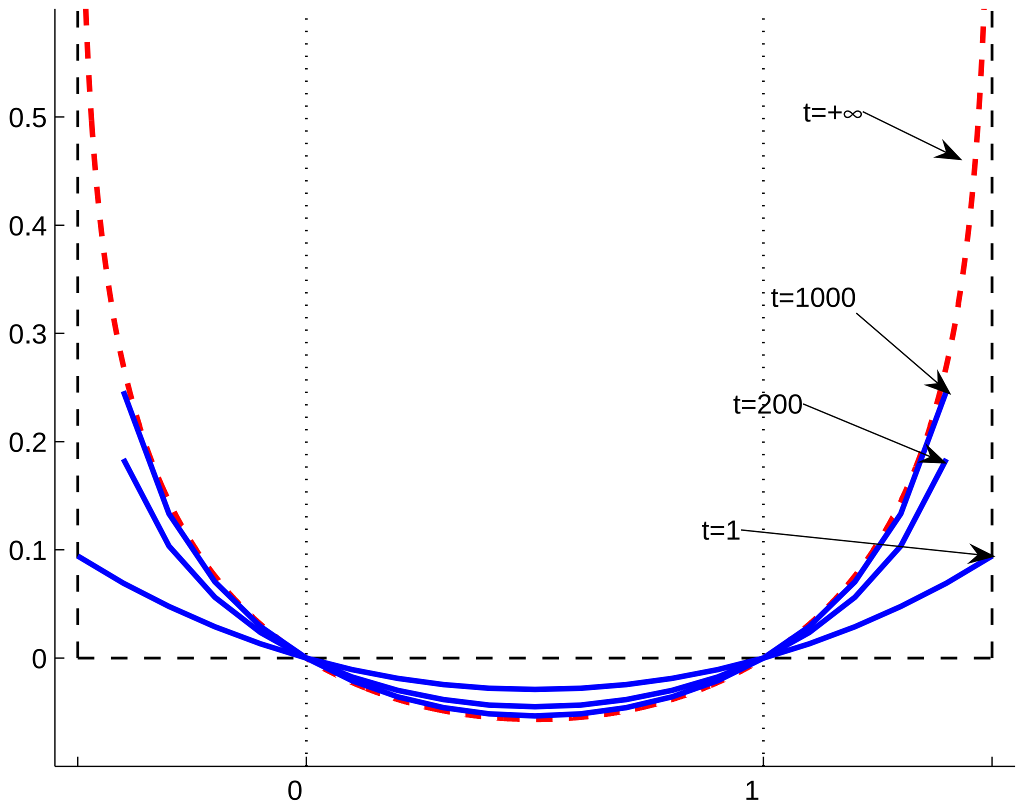
- $e_2(\alpha)$ is convex, $e_2(0) = e_2(1) = 0$, $e_2(\alpha) \le 0$ for $\alpha \in [0,1]$, TRI $\Rightarrow e_2(\alpha) = e_2(1-\alpha)$
- $\langle \sigma \rangle_+ = \lim_{t\to\infty} \omega(\Sigma^t) = -\lim_{t\to\infty} \frac{1}{t} S(\omega_t|\omega)$ asymptotic mean entropy production rate
- The large deviation principle

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_t(\mathbb{J}) = -\inf_{\theta\in\mathbb{J}} I(\theta)$$

holds for any open $\mathbb{J} \subset]-\bar{\theta}, \bar{\theta}[$ ($\bar{\theta} = \sup_{\alpha\in\mathbb{I}} e_2'(\alpha)$) with rate function

$$I(\theta) = -\inf_{\alpha\in\mathbb{I}} (e_2(\alpha) + \theta\alpha)$$

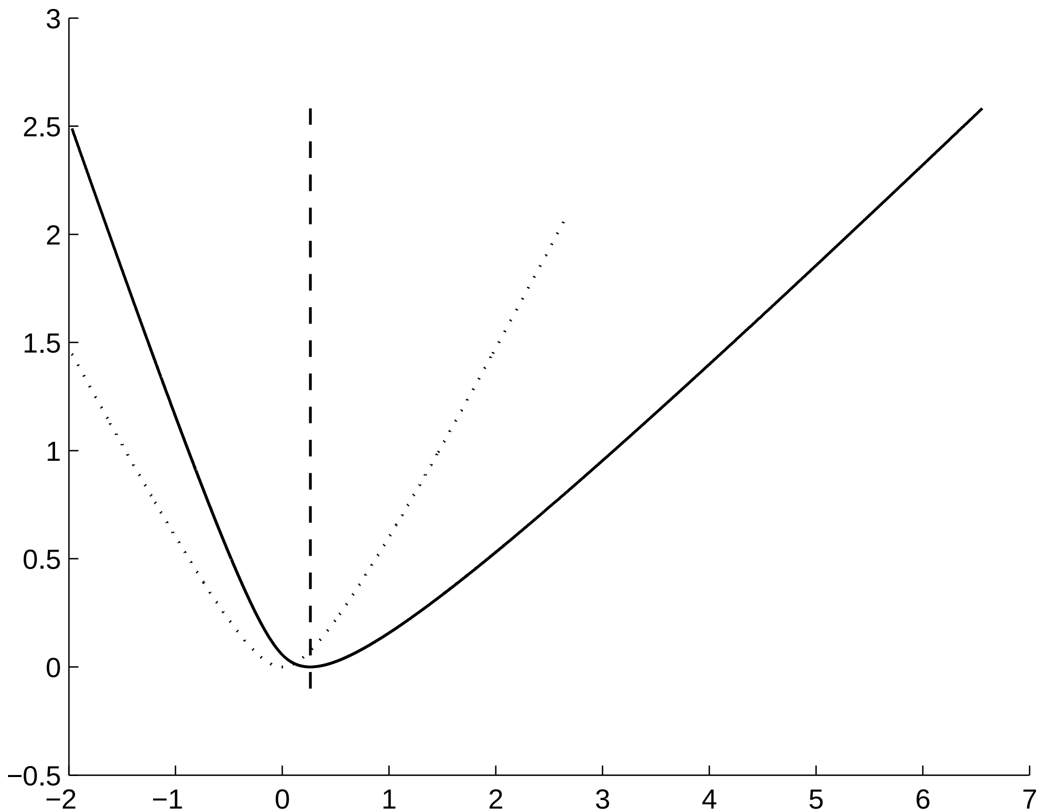
LARGE TIME LIMIT



LARGE TIME LIMIT

The Evans-Searles symmetry $e_2(\alpha) = e_2(1-\alpha)$ translates into

$$I(-\theta) = I(\theta) + \theta$$



QUANTUM HYPOTHESIS TESTING

- Von Neumann algebra $\mathfrak{M}$ on Hilbert space $\mathcal{H}$. Let $\rho \neq \nu$ be two faithful normal states of the quantum system described by $\mathfrak{M}$. A priory the system could be in either state with probability $1/2$. By performing a measurement we wish to decide with minimal error probability the true state of the system. A test T is an orthogonal projection in $\mathfrak{M}$ and the results of its measurement is a number in $\mathrm{sp}(T) = \{0, 1\}$. If 1 is observed, one accepts ρ, otherwise one accepts ν.
- Error probability for a given test T

$$D(\rho, \nu, T) = \frac{1}{2}\rho(I - T) + \frac{1}{2}\nu(T)$$

- Minimal error probability

$$D(\rho, \nu) = \inf_T D(\rho, \nu, T)$$

- Quantum Nayman-Pearson lemma: optimal test exists and is unique

$$T_{\mathrm{opt}} = \text{support projection of } (\nu - \rho)_+, \qquad D(\rho, \nu) = \frac{1}{2}\left(1 - \frac{1}{2}\|\rho - \nu\|\right)$$

QUANTUM HYPOTHESIS TESTING

- Lower bound: let $\mu_{\rho|\nu}$ be the spectral measure of $\log \Delta_{\rho|\nu}$ for Ω_ν, then

$$\frac{1}{4}\mu_{\rho|\nu}(]0,\infty[) \leq D(\rho,\nu)$$

easy to prove!

- Chernoff upper bound: for any $s \in [0,1]$,

$$D(\rho,\nu) \leq \frac{1}{2}(\Omega_\nu|\Delta^s_{\rho|\nu}\Omega_\nu)$$

  - For $s = 1/2 \Leftrightarrow \|\Omega_\rho - \Omega_\nu\|^2 \leq \|\rho - \nu\|$: [Araki 1974].
  - For finite dimensional $\mathfrak{M} \Leftrightarrow \frac{1}{2}\mathrm{Tr}\,(A + B - |A - B|) \leq \mathrm{Tr}\,A^{1-s}B^s$, for positive matrices A, B. [Audenaert et al., PRL+CMP 2008]. Tricky and convoluted, impossible to extend to infinite dimensional case.
  - New, short and completely elementary proof: [Ozawa 2010, unpublished].
  - Extended to general von Neumann algebras by Ogata (2010) using Connes cocycles and by Jakšić, Ogata, P., Seiringer (2011) using non-commutative L^p-spaces.

ASYMPTOTIC QUANTUM HYPOTHESIS TESTING

- Consider a family $(\mathfrak{M}_t, \rho_t, \nu_t)_{t>0}$ and define
  - Chernoff exponents

$$\underline{D} = \liminf_t \frac{1}{t} \log D(\rho_t, \nu_t), \qquad \overline{D} = \limsup_t \frac{1}{t} \log D(\rho_t, \nu_t)$$

  - Hoefding exponents ($r > 0$)

$$B(r) = \inf_{\{T_t\}} \left\{ \lim_t \frac{1}{t} \log \rho_t(I - T_t) \,\middle|\, \limsup_t \frac{1}{t} \log \nu_t(T_t) < -r \right\}$$

  - Stein exponents ($\epsilon \in]0, 1[$)

$$B_\epsilon = \inf_{\{T_t\}} \left\{ \lim_t \frac{1}{t} \log \rho_t(I - T_t) \,\middle|\, \nu_t(T_t) < \epsilon \right\}$$

ASYMPTOTIC QUANTUM HYPOTHESIS TESTING

- Assuming: There is $\delta > 0$ s.t.

$$e(\alpha) = \lim_t \frac{1}{t} \log(\Omega_{\nu_t} | \Delta^{\alpha}_{\rho_t|\nu_t} \Omega_{\nu_t}) = \lim_t \frac{1}{t} S_\alpha(\rho_t|\nu_t)$$

exists and is a differentiable function of $\alpha \in]-\delta, 1+\delta[$, with $e'(0) < 0$.

- One has:
  - $\underline{D} = \overline{D} = \inf_{\alpha \in [0,1]} e(\alpha)$
  - $B(r) = \inf_{\alpha \in]0,1[} \frac{\alpha r + e(\alpha)}{1-\alpha}$
  - $B_\epsilon = e'(0)$

- Given the lower bound and the Chernoff upper bound, proofs of these facts are identical to the proofs of the corresponding classical results.

EXAMPLE: SPIN SYSTEMS

- $\Lambda \subset \mathbb{Z}^d$, $t = |\Lambda|$, interactions Ψ, Φ, Gibbs state

$$\rho_\Lambda = \frac{\mathrm{e}^{-H_\Lambda(\Psi)}}{\mathrm{Tr}\,\mathrm{e}^{-H_\Lambda(\Psi)}}, \qquad \nu_\Lambda = \frac{\mathrm{e}^{-H_\Lambda(\Phi)}}{\mathrm{Tr}\,\mathrm{e}^{-H_\Lambda(\Phi)}},$$

- $$e(\alpha) = \lim_\Lambda \frac{1}{|\Lambda|} \log \mathrm{Tr}\left(\mathrm{e}^{-\alpha H_\Lambda(\Phi)}\mathrm{e}^{-(1-\alpha)H_\Lambda(\Psi)}\right) - \alpha P(\Phi) - (1-\alpha)P(\Psi)$$

  - [Ogata 2010] $d = 1$, short range: $e(\alpha)$ is real analytic on $\mathbb{R}$.
  - [Netocny–Redig 2004, Lenci–Rey-Bellet 2005] $d > 1$, finite range, high temperature: $e(\alpha)$ is real analytic on $]-r, r[$.
  - $e(\alpha)$ is strictly convex iff Ψ and Φ are not physically equivalent.

TESTING TIME'S ARROW

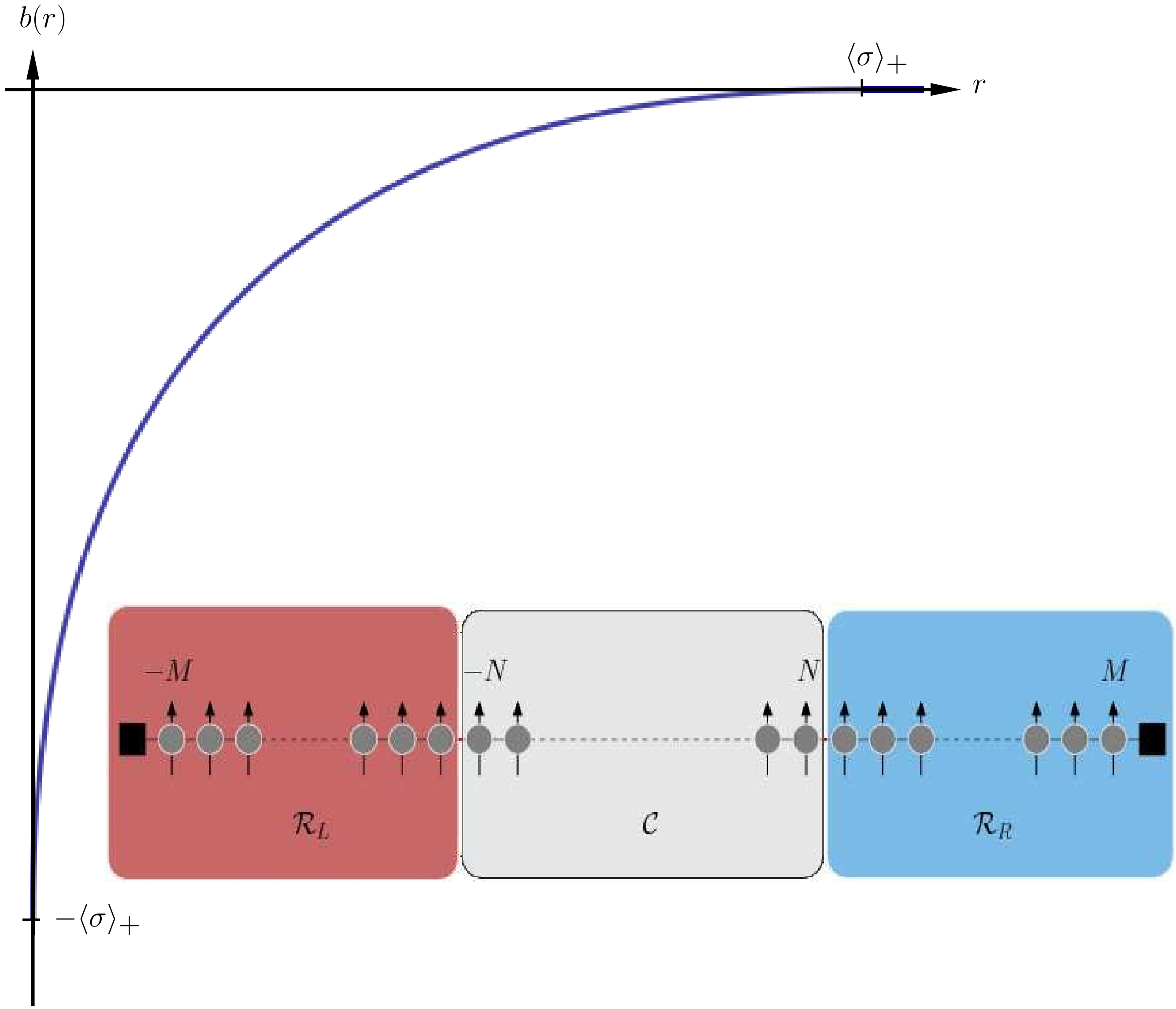
- Compare the definition of the FCS entropic functional

$$e_{2,t}(\alpha) = S_\alpha(\omega_t|\omega) = \log \mathrm{Tr}\, \left(\omega_t^\alpha \omega^{1-\alpha}\right) = \log \mathrm{Tr}\, \left(\omega_{t/2}^\alpha \omega_{-t/2}^{1-\alpha}\right) = S_\alpha(\omega_{t/2}|\omega_{-t/2})$$

  with asymptotic hypothesis testing $\Longrightarrow$

- The entropic functional $e_2(\alpha) = \lim_{t\to\infty} t^{-1} e_{2,t}(\alpha)$ which describes the large time asymptotics of FCS also controls asymptotic QHT for the family of states $\{\omega_t, \omega\}_{t>0} \sim \{\omega_{t/2}, \omega_{-t/2}\}_{t>0}$, i.e., distinguishing the future from the past.
- Strictly positive entropy production $\langle\sigma\rangle_+ > 0 \Rightarrow \omega_t$ and ω become mutually singular: their supports become orthogonal ($\omega \perp \rho \Leftrightarrow \|\omega - \rho\| = 2$)
  - Chernoff exponent: $2 \geq \|\omega_t - \omega\| \simeq 2(1 - \mathrm{e}^{-Dt})$, $D = -\inf_{\alpha\in[0,1]} e_2(\alpha) > 0$
  - Stein exponent: for any $\epsilon, \delta > 0$ there is $\{T_t\}$ such that $\sup_t \omega(T_t) < \epsilon$ and $1 - \mathrm{e}^{-t(\langle\sigma\rangle_+ - \delta)} \leq \omega_t(T_t) \leq 1 - \mathrm{e}^{-t\langle\sigma\rangle_+}$, i.e., $\langle\sigma\rangle_+$ is the maximal concentration rate of ω_t out of the support of ω
  - Hoefding exponent: for any $r > 0$ there is $\{T_t\}$ such that $\omega(T_t) \leq \mathrm{e}^{-tr}$ and $\omega_t(T_t) \sim 1 - \mathrm{e}^{tb(r)}$

TESTING TIME'S ARROW



OUTLOOK

- Physical interpretation of Tasaki-Matsui spectral fluctuation relation in terms of FCS (extends to multiparameter case – individual fluxes).
- Large Deviation Principle (and Central Limit Theorem) for FCS:
  - Easy to implement for quasi-free systems, e.g., Landauer-Büttiker transport theory.
  - For Pauli-Fierz models: alternative to [De Roeck 2009] through spectral analysis of L^p-Liouvillians.
- General scheme for Quantum Hypothesis Testing in von Neumann algebras.
- Concentration estimates for approach to NESS.
- New problems:
  - Multi-parameter FCS $\Leftrightarrow$ Multi-state discrimination ?
  - $e(\alpha)$ for low temperature quantum spin systems.
  - Quantum information interpretation of more general entropic functionals

$$e_{p,t}(\alpha) = \log \mathrm{Tr} \left(\omega^{(1-\alpha)/p} \omega_t^{2\alpha/p} \omega^{(1-\alpha)/p} \right)^{p/2}$$

Proof of Nayman-Pearson lemma. For any test T,

$$\rho(1-T)+\nu(T)=1-(\rho-\nu)(T)\geq 1-(\rho-\nu)_+(T)\left[=1-\frac{1}{2}((\rho-\nu)(T)+|\rho-\nu|(T))\right]$$

$$\geq 1-(\rho-\nu)_+(1)=1-\frac{1}{2}\|\rho-\nu\|$$

on the other hand, for $T=\mathrm{s}_{(\rho-\nu)_+}$

$$\rho(1-T)+\nu(T)=1-(\rho-\nu)_+(1)=1-\frac{1}{2}\|\rho-\nu\|$$

Proof of Lower bound. If S and $P \geq 0$ are bounded self-adjoint operators then

$$SPS + (1-S)P(1-S) = \frac{1}{2}P + \frac{1}{2}(1-2S)P(1-2S) \geq \frac{1}{2}P$$

With $S = \mathrm{s}_{(\rho-\nu)_+}$

$$\nu(S) = (S\Omega_\nu|S\Omega_\nu) = (\Delta_{\nu|\rho}^{1/2}S\Omega_\rho|\Delta_{\nu|\rho}^{1/2}S\Omega_\rho),$$

and

$$\begin{aligned}
\rho(1-S) + \nu(S) &= (\Omega_\rho|(1 - S + S\Delta_{\nu|\rho}S)\Omega_\rho) \\
&= \int_0^\infty (\Omega_\omega\,|[(1-S)\mathrm{d}P(\lambda)(1-S) + \lambda S\mathrm{d}P(\lambda)S]\;\Omega_\omega) \\
&\geq \int_1^\infty (\Omega_\omega\,|[(1-S)\mathrm{d}P(\lambda)(1-S) + S\mathrm{d}P(\lambda)S]\;\Omega_\omega) \\
&\geq \frac{1}{2}\int_1^\infty (\Omega_\omega|\mathrm{d}P(\lambda)\Omega_\omega) = \frac{1}{2}\,\mu_{\nu|\rho}([1,\infty[),
\end{aligned}$$

Proof of Chernoff upper bound [Ozawa 2010, unpublished]. One has to show that for any positive matrices A, B and $s \in [0,1]$,

$$\frac{1}{2}\mathrm{Tr}\ (A + B - |A - B|) \leq \mathrm{Tr}\ \left(A^{1-s} B^s\right)$$

which is equivalent to

$$\mathrm{Tr}\ \left(A - A^{1-s} B^s\right) \leq \mathrm{Tr}\,(A - B)_+$$

Since $B + (A - B)_+ \geq B$ and $B + (A - B)_+ = A + (A - B)_- \geq A$, it follows from the operator monotonicity of $x \mapsto x^s$

$$\begin{aligned}
\mathrm{Tr}\ \left(A - A^{1-s} B^s\right) &= \mathrm{Tr}\ \left(A^{1-s}(A^s - B^s)\right) \\
&\leq \mathrm{Tr}\ \left(A^{1-s}((B + (A - B)_+)^s - B^s)\right) \\
&\leq \mathrm{Tr}\ \left((B + (A - B)_+)^{1-s}((B + (A - B)_+)^s - B^s)\right) \\
&= \mathrm{Tr}\ \left(B + (A - B)_+ - ((B + (A - B)_+)^{1-s} B^s\right) \\
&\leq \mathrm{Tr}\ \left(B + (A - B)_+ - B^{1-s} B^s\right) = \mathrm{Tr}\,(A - B)_+
\end{aligned}$$