Quantum Koopmanism

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The Koopman-von Neumann spectral approach to ergodic theory is a powerful tool in the study of statistical properties of dynamical systems (see [Ergodic theory. Mixing], [Spectrum of dynamical system]). Its extension to quantum dynamical systems – the spectral theory of Liouvilleans – is at the center of many recent results in quantum statistical mechanics (see [Quantum nonequilibrium statistical mechanics], [Return to equilibrium], [NESS in quantum statistical mechanics] as well as [BFS],[DJ], [FM], [JP]).

Let $(\mathcal{O},\tau)$ be a C^*- or W^*-dynamical system equipped with a τ-invariant state ω, assumed to be normal in the W^*-case. The GNS-representation $(\mathcal{H}_\omega,\pi_\omega,\Omega_\omega)$ maps the triple $(\mathcal{O},\tau,\omega)$ into $(\mathcal{O}_\omega,\tilde\tau,\tilde\omega)$, a W^*-dynamical system on the enveloping von Neumann algebra $\mathcal{O}_\omega=\pi_\omega(\mathcal{O})''$ with a normal invariant state $\tilde\omega(A)=(\Omega_\omega|A\Omega_\omega)$. The W^*-dynamics $\tilde\tau$ is given by

$$\tilde\tau^t(A)=\mathrm{e}^{\mathrm{i}tL_\omega}A\mathrm{e}^{-\mathrm{i}tL_\omega},$$

where L_ω is the ω-Liouvillean (see Section 3 in [Quantum dynamical systems]).

We shall say that $(\pi_\omega,\mathcal{O}_\omega,\mathcal{H}_\omega,L_\omega,\Omega_\omega)$ is the normal form of $(\mathcal{O},\tau,\omega)$.

1 Ergodic properties of quantum dynamical systems

Let $\mathfrak{M}$ be a von Neumann algebra acting on the Hilbert space $\mathcal{H}$. The support s_ω of a normal state ω on $\mathfrak{M}$ is the smallest orthogonal projection $P\in\mathfrak{M}$ such that $\omega(P)=1$. A normal state ω is faithful if and only if $s_\omega=I$. The support of the state $\omega(A)=(\Omega|A\Omega)$ is the orthogonal projection on the closure of the subspace $\mathfrak{M}'\Omega$.

Notation. We write $\nu\ll\omega$ whenever ν is a ω-normal state such that $s_\nu\le s_\omega$.

Remark. If $\mathfrak{M}$ is Abelian any ω-normal state ν satisfies $\nu\ll\omega$. This explains why the support condition is absent in classical ergodic theory (the reader may consult [P] for a detailed discussion of this point). In most applications to statistical mechanics ω is faithful and any ω-normal state ν satisfies $\nu\ll\omega$.

Definition 1 *Let $(\mathfrak{M},\tau)$ be a W^*-dynamical system and ω a normal τ-invariant state.*

1. *$(\mathfrak{M},\tau,\omega)$ is ergodic if*

   $$\lim_{T\to\infty}\frac{1}{T}\int_0^T\omega(As_\omega\tau^t(B))\,\mathrm{d}t=\omega(A)\omega(B),$$

   holds for any $A,B\in\mathcal{O}$.

2. *$(\mathfrak{M},\tau,\omega)$ is weakly mixing if*

   $$\lim_{T\to\infty}\frac{1}{T}\int_0^T\left|\omega(As_\omega\tau^t(B))-\omega(A)\omega(B)\right|\,\mathrm{d}t=0,$$

   holds for all $A,B\in\mathcal{O}$.

3. $(\mathfrak{M}, \tau, \omega)$ *is mixing or returns to equilibrium if*

$$\lim_{t\to\infty} \omega(A s_\omega \tau^t(B)) = \omega(A)\omega(B),$$

for any $A, B \in \mathcal{O}$

4. *If* ω *is an invariant state of the* C^**-dynamical system* $(\mathcal{O}, \tau)$ *we say that* $(\mathcal{O}, \tau, \omega)$ *is ergodic (resp. mixing, weakly mixing) if* $(\mathcal{O}_\omega, \tilde{\tau}, \tilde{\omega})$ *is ergodic (resp. mixing, weakly mixing).*

Remark. Ergodicity 1 is equivalent to

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \nu(\tau^t(A))\, \mathrm{d}t = \omega(A),$$

for all $A \in \mathfrak{M}$ and all states $\nu \ll \omega$. The mixing property 3 is equivalent to

$$\lim_{t\to\infty} \nu(\tau^t(A)) = \omega(A),$$

for all $A \in \mathfrak{M}$ and all states $\nu \ll \omega$ (see [R], [JP]).

2 Spectral characterization of ergodic properties

We refer to [P] for proofs of the results in this section.

The following theorem is the quantum version of the well known Koopman-von Neumann spectral characterizations ([AA], [K], [N]).

Theorem 2 *Let* $(\mathcal{O}, \tau)$ *be a* C^**- or* W^**-dynamical system equipped with a* τ*-invariant state* ω*, assumed to be normal in the* W^**-case. Denote by* $(\pi_\omega, \mathcal{O}_\omega, \mathcal{H}_\omega, L_\omega, \Omega_\omega)$ *its normal form and by* $\mathcal{K}_\omega$ *the closure of* $\pi_\omega(\mathcal{O})'\Omega_\omega$*.*

1. *The subspace* $\mathcal{K}_\omega$ *reduces the operator* L_ω*. Denote by* $\mathfrak{L}_\omega$ *the restriction* $L_\omega\big|_{\mathcal{K}_\omega}$*.*

2. $(\mathcal{O}, \tau, \omega)$ *is ergodic if and only if* $\mathrm{Ker}(\mathfrak{L}_\omega)$ *is one dimensional.*

3. $(\mathcal{O}, \tau, \omega)$ *is weakly mixing if and only if* 0 *is the only eigenvalue of* $\mathfrak{L}_\omega$ *and* $\mathrm{Ker}(\mathfrak{L}_\omega)$ *is one dimensional.*

4. $(\mathcal{O}, \tau, \omega)$ *is mixing if and only if*
$$\mathrm{w}-\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}t\mathfrak{L}_\omega} = \Omega_\omega(\Omega_\omega|\,\cdot\,).$$

5. *If the spectrum of* $\mathfrak{L}_\omega$ *on* $\{\Omega_\omega\}^\perp$ *is purely absolutely continuous then* $(\mathcal{O}, \tau, \omega)$ *is mixing.*

Note that $\mathcal{K}_\omega$ is the range of the support of $\tilde{\omega}$. Thus, if $\tilde{\omega}$ is faithful then $\mathfrak{L}_\omega = L_\omega$.

Like the classical Koopman operator, the reduced Liouvillean $\mathfrak{L}_\omega$ of an ergodic quantum dynamical system has a number of peculiar spectral properties.

Theorem 3 *Assume, in addition to the hypotheses of the previous theorem, that* $(\mathcal{O}, \tau, \omega)$ *is ergodic. Then the following hold:*

1. *The point spectrum of* $\mathfrak{L}_\omega$ *is a subgroup* Σ *of the additive group* $\mathbb{R}$*.*

2. *The eigenvalues of* $\mathfrak{L}_\omega$ *are simple.*

3. *The spectrum of* $\mathfrak{L}_\omega$ *is invariant under translations in* Σ*, that is,* $\mathrm{spec}(\mathfrak{L}_\omega) + \Sigma = \mathrm{spec}(\mathfrak{L}_\omega)$.

4. *If* Ψ *is a normalized eigenvector of* $\mathfrak{L}_\omega$ *then* $(\Psi|\pi_\omega(A)\Psi) = \omega(A)$ *for all* $A \in \mathcal{O}$*.*

5. *If* $(\mathcal{O}, \tau, \omega)$ *is mixing then* 0 *is the only eigenvalue of* $\mathfrak{L}_\omega$*.*

6. *If* ω *is a* (τ, β)*-<u>KMS state</u> then* $\Sigma = \{0\}$ *and the system is weakly mixing.*

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