Quantum dynamical systems

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In the most common approach to quantum physics observables are described by all operators on a certain Hilbert space. This formalism is usually sufficient in the case of zero temperature. To describe quantum systems in thermodynamic limit at positive temperatures, or more generally, at positive densities, it is convenient to use a more sophisticated formalism where observables are described by elements of some operator algebra (see [The C^*-algebra approach], [Free Bose and Fermi gases – the algebraic approach]). In this algebraic approach, quantum dynamics in the Heisenberg picture is given by a one-parameter group of automorphisms of the algebra of observables. In analogy with the theory of classical dynamical systems, one says that such a group defines a quantum dynamical system. There are two versions of the algebraic approach. They differ in the topological properties of the algebra and of the dynamical group: the C^*- and the W^*-approach.

1 C^*-dynamical systems

Definition 1 *A C^*-dynamics on a C^*-algebra $\mathcal{O}$ is a strongly continuous one-parameter group of $*$-automorphisms of $\mathcal{O}$, $\mathbb{R} \ni t \to \tau^t$. A C^*-dynamical system is a pair $(\mathcal{O}, \tau)$ where $\mathcal{O}$ is a C^*-algebra and τ a C^*-dynamics on $\mathcal{O}$.*

The strong continuity of τ means that the map $t \mapsto \tau^t(A)$ is norm-continuous for any $A \in \mathcal{O}$. From the general theory of strongly continuous groups on a Banach space, a C^*-dynamics τ has a densely defined, closed infinitesimal generator δ such that

$$\delta(A) = \lim_{t\to 0} \frac{\tau^t(A) - A}{t}, \tag{1}$$

for $A \in \mathrm{Dom}\,\delta$. In particular, if $\mathcal{O}$ has a unit I then $I \in \mathrm{Dom}\,\delta$ and $\delta(I) = 0$. One easily sees that δ is a $*$-derivation i.e., that

1. $\mathrm{Dom}\,\delta$ is a $*$-algebra.
2. $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in \mathrm{Dom}\,\delta$.
3. $\delta(A^*) = \delta(A)^*$ for all $A \in \mathrm{Dom}\,\delta$.

Generators of C^*-dynamics are characterized by the following simple adaptation of the Hille-Yosida Theorem (see [BR1]).

Theorem 2 *Let $\mathcal{O}$ be a C^*-algebra. A norm densely defined and closed operator δ on $\mathcal{O}$ generates a C^*-dynamics if and only if*

1. *δ is a $*$-derivation.*
2. *$\mathrm{Ran}(\mathrm{Id} + \lambda\delta) = \mathcal{O}$ for all $\lambda \in \mathbb{R}$.*
3. *$\|A + \lambda\delta(A)\| \geq \|A\|$ for all $\lambda \in \mathbb{R}$ and $A \in \mathrm{Dom}\,\delta$.*

Example 1. Let $\mathcal{H}$ be a Hilbert space and H a bounded self-adjoint operator on $\mathcal{H}$. Then $\tau^t(A) = \mathrm{e}^{\mathrm{i}tH}A\mathrm{e}^{-\mathrm{i}tH}$ is a C^*-dynamics on $\mathcal{B}(\mathcal{H})$. Its generator $\delta(A) = \mathrm{i}[H, A]$ is bounded. Note that boundedness of H is required for $t \to \tau^t$ to be strongly continuous in this case.

Example 2. The following is based on the material of Subsection 2.2 in [The C^*-algebra approach]. Let $\mathfrak{h}$ be a Hilbert space and h a self-adjoint operator on $\mathfrak{h}$. The group of Bogoliubov automorphisms of the C^*-algebra $\mathrm{CAR}(\mathfrak{h})$ defined by $\tau^t(a(f)) = a(\mathrm{e}^{\mathrm{i}th}f)$ is a C^*-dynamics. This is a consequence of the strong continuity of the unitary group $t \mapsto \mathrm{e}^{\mathrm{i}th}$ and of the norm continuity of the map $f \mapsto a(f)$ from $\mathfrak{h}$ to $\mathrm{CCR}(\mathfrak{h})$. The $*$-subalgebra generated by $\{a(f) \,|\, f \in \mathrm{Dom}\, h\}$ is contained in the domain of the generator δ of τ and $\delta(a(f)) = a(\mathrm{i}hf)$.

2 W^*-dynamical systems

In some cases (e.g. for systems of bosons), the C^*-approach is not adequate and one has to use the W^* setting.

Definition 3 *Let $\mathfrak{M}$ be a von Neumann algebra or a W^*-algebra. A W^*-dynamics on $\mathfrak{M}$ is an σ-weakly continuous group $\mathbb{R} \ni t \mapsto \tau^t$ of $*$-automorphisms of $\mathfrak{M}$. A W^*-dynamical system is a pair $(\mathfrak{M}, \tau)$ where $\mathfrak{M}$ is a von Neumann algebra and τ a W^*-dynamics on $\mathfrak{M}$.*

The continuity condition on the group τ means that, for any $A \in \mathfrak{M}$, the map $t \mapsto \tau^t(A)$ is continuous in the σ-weak topology of $\mathfrak{M}$. The generator δ of a W^*-dynamics τ on a von Neumann algebra $\mathfrak{M}$ is defined by Equ. (1), as in the C^*-case, except that the limit is now understood in the σ-weak topology. It is a σ-weakly densely defined and closed $*$-derivation on $\mathfrak{M}$ such that $I \in \mathrm{Dom}\,\delta$ and $\delta(I) = 0$. Generators of W^*-dynamics are characterized by the following analog of Theorem 2 (see [BR1]).

Theorem 4 *Let $\mathfrak{M}$ be a von Neumann algebra. A σ-weakly densely defined and closed operator δ on $\mathfrak{M}$ generates a W^*-dynamics if and only if*

1. *δ is a $*$-derivation and $I \in \mathrm{Dom}\,\delta$.*
2. *$\mathrm{Ran}(\mathrm{Id} + \lambda\delta) = \mathcal{O}$ for all $\lambda \in \mathbb{R}$.*
3. *$\|A + \lambda\delta(A)\| \geq \|A\|$ for all $\lambda \in \mathbb{R}$ and $A \in \mathrm{Dom}\,\delta$.*

Example 3. Let $\mathcal{H}$ be a Hilbert space and H a self-adjoint operator on $\mathcal{H}$. Then $\tau^t(A) = \mathrm{e}^{\mathrm{i}tH}A\mathrm{e}^{-\mathrm{i}tH}$ is a W^*-dynamics on $\mathcal{B}(\mathcal{H})$.

Example 4. The following is based on the material of Subsection 2.3 in [The C^*-algebra approach]. Le $\mathfrak{h}$ be a Hilbert space and h a self-adjoint operator on $\mathfrak{h}$. Denote by $\mathfrak{h}_0$ a subspace of $\mathfrak{h}$ invariant under the unitary group $\mathrm{e}^{\mathrm{i}th}$. Except in trivial cases the group of Bogoliubov automorphisms of the C^*-algebra $\mathrm{CCR}(\mathfrak{h}_0)$ defined by

$$\tau^t(W(f)) = W(\mathrm{e}^{\mathrm{i}th}f),$$

does not define a C^*-dynamics because $\|\tau^t(W(f)) - W(f)\| = 2$ for all $t \in \mathbb{R}$ and $f \in \mathfrak{h}_0$ such that $\mathrm{e}^{\mathrm{i}th}f \neq f$.

Denote by $\mathfrak{M}$ the von Neumann algebra acting on the bosonic Fock space $\Gamma_{\mathrm{s}}(\mathfrak{h})$ and generated by the Weyl operators $\{W(f) \,|\, f \in \mathfrak{h}_0\}$. Then τ has an extension to $\mathfrak{M}$ given by

$$\tau^t(A) = \mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(h)}A\mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma(h)}.$$

It defines a W^*-dynamics on $\mathfrak{M}$.

Example 3. See [Free Bose and Fermi gases – the algebraic approach].

3 Invariant states and Liouvilleans

In this and the following section we use freely the notations of Section 3 in [The C^*-algebra approach]. We shall say that $(\mathcal{O}, \tau)$ is a quantum dynamical system if it is either a C^*- or a W^*-dynamical system.

Definition 5 *Let $(\mathcal{O}, \tau)$ be a quantum dynamical system. A state ω on $\mathcal{O}$ is τ-invariant if $\omega \circ \tau^t = \omega$ holds for all $t \in \mathbb{R}$.*

As in the theory of classical dynamical systems, invariant states (and more specifically normal invariant states in the W^*-case) play an important role in the analysis of quantum dynamical systems. As an illustration let us explore the GNS representation induced by an invariant state.

Let ω be an invariant state of the quantum dynamical system $(\mathcal{O}, \tau)$ and denote by $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ the GNS representation of $\mathcal{O}$ induced by ω. The unicity of the GNS representation implies that there exists a unique one-parameter group $t \mapsto U_\omega(t)$ of unitary operators on $\mathcal{H}_\omega$ such that

$$\pi_\omega(\tau^t(A)) = U_\omega(t)\pi_\omega(A)U_\omega(t)^*, \quad U_\omega(t)\Omega_\omega = \Omega_\omega,$$

for any $t \in \mathbb{R}$ and $A \in \mathcal{O}$. Assuming ω to be normal in the W^*-case, it is easy to show that the group U_ω is strongly continuous. Hence, by Stone's theorem, there exists a unique self-adjoint operator L_ω such that

$$\mathrm{e}^{\mathrm{i}tL_\omega}\pi_\omega(A)\mathrm{e}^{-\mathrm{i}tL_\omega} = \pi_\omega(\tau^t(A)), \quad L_\omega\Omega_\omega = 0.$$

The operator L_ω is sometimes called ω-Liouvillean of $(\mathcal{O}, \tau)$. Important information on the dynamics of the system can be deduced from its spectral properties, see [Tomita-Takesaki theory], [Spectral analysis of small quantum systems interacting with a reservoir], [Quantum Koopmanism] and [Return to equilibrium].

Next we note that

$$\tilde{\omega}(A) = (\Omega_\omega | A\Omega_\omega),$$

defines a normal extension of the state ω to the enveloping von Neumann algebra $\mathcal{O}_\omega$: $\omega = \tilde{\omega} \circ \pi_\omega$. Similarly,

$$\tilde{\tau}^t(A) = \mathrm{e}^{\mathrm{i}tL_\omega} A \mathrm{e}^{-\mathrm{i}tL_\omega},$$

defines a W^*-dynamics on $\mathcal{O}_\omega$ such that $\tilde{\tau}^t \circ \pi_\omega = \pi_\omega \circ \tau^t$. We conclude that the GNS construction maps a C^*-dynamical system $(\mathcal{O}, \tau)$ with invariant state ω into a W^*-dynamical system $(\mathcal{O}_\omega, \tilde{\tau})$ with normal invariant state $\tilde{\omega}$.

The above construction can be performed under weaker continuity conditions that the strong/σ-weak continuity used here, see [P] for a more general definition of quantum dynamical systems.

4 Perturbation theory

Let $(\mathcal{O}, \tau)$ be a C^*-dynamical system and $V \in \mathcal{O}$ a self-adjoint element. If δ denotes the generator of τ then

$$\delta_V = \delta + \mathrm{i}[V, \cdot\,],$$

is well defined on $\mathrm{Dom}\,\delta$ and generates a perturbed dynamics $\tau_V^t = \mathrm{e}^{t\delta_V}$ on $\mathcal{O}$. One says that τ_V is a local perturbation of τ. Such local perturbations play an important role in the theory of C^*-dynamical systems.

Iterating the integral equation (Duhamel formula)

$$\tau_V^t(A) = \tau^t(A) + \int_0^t \tau^{t-s}(\mathrm{i}[V, \tau_V^s(A)])\,\mathrm{d}s,$$

leads to the Araki-Dyson expansion

$$\tau_V^t(A) = \tau^t(A) + \sum_{n=1}^{\infty} \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \, \mathrm{i}[\tau^{t_n}(V), \mathrm{i}[\cdots, \mathrm{i}[\tau^{t_1}(V), \tau^t(A)] \cdots]],$$

which is norm convergent for any $t \in \mathbb{R}$ and $A \in \mathcal{O}$. Another useful representation of the locally perturbed dynamics is the interaction picture $\tau_V^t(A) = \Gamma_V^t \tau^t(A) \Gamma_V^{t*}$. The operator Γ_V^t is the solution of the differential equation

$$\partial_t \Gamma_V^t = \mathrm{i}\Gamma_V^t \tau^t(V),$$

with the initial condition $\Gamma_V^0 = I$. It follows that $\Gamma_V^t \in \mathcal{O}$ is unitary and has the norm convergent Dyson expansion

$$\Gamma_V^t = I + \sum_{n=1}^{\infty} \mathrm{i}^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \, \tau^{t_n}(V) \cdots \tau^{t_1}(V).$$

Moreover it satisfies the cocycle relation

$$\Gamma_V^{t+s} = \Gamma_V^t \tau^t(\Gamma_V^s) = \tau_V^t(\Gamma_V^s)\Gamma_V^t.$$

Local perturbations of W^*-dynamical systems can be handled in a similar way, replacing the norm topology with the σ-weak topology and interpreting all integrals in the weak-$*$ sense.

If ω is an invariant state for the unperturbed dynamical system $(\mathcal{O}, \tau)$ (supposed to be normal in the W^*-case) then, in the induced GNS representation and with the notation of the previous subsection, the perturbed dynamics is implemented by the unitary group generated by $L_\omega + Q$ where $Q = \pi_\omega(V)$,

$$\pi_\omega(\tau_V^t(A)) = \mathrm{e}^{\mathrm{i}t(L_\omega+Q)} \pi_\omega(A) \mathrm{e}^{-\mathrm{i}t(L_\omega+Q)}.$$

Note that the perturbed unitary group is related to the cocycle Γ_V through the interaction picture formula

$$\mathrm{e}^{\mathrm{i}t(L_\omega+Q)} = \widetilde{\Gamma}_V^t \, \mathrm{e}^{\mathrm{i}tL_\omega}, \quad \widetilde{\Gamma}_V^t = \pi_\omega(\Gamma_V^t) = I + \sum_{n=1}^{\infty} \mathrm{i}^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \, \tilde{\tau}^{t_n}(Q) \cdots \tilde{\tau}^{t_1}(Q).$$

Consequently, the perturbed dynamics extends to a W^*-dynamics

$$\tilde{\tau}_V^t(A) = \mathrm{e}^{\mathrm{i}t(L_\omega+Q)} A \mathrm{e}^{-\mathrm{i}t(L_\omega+Q)} = \widetilde{\Gamma}_V^t \tilde{\tau}^t(A) \widetilde{\Gamma}_V^{t*},$$

on the enveloping von Neumann algebra $\mathcal{O}_\omega$. In the W^*-case this formula is the starting point for an extension of perturbation theory to unbounded perturbations. If Q is a selfadjoint operator on $\mathcal{H}_\omega$ affiliated to $\mathcal{O}_\omega = \pi_\omega(\mathcal{O})$ and such that $L_\omega + Q$ is essentially self-adjoint on $\mathrm{Dom}(L_\omega) \cap \mathrm{Dom}(Q)$ then the unitary group $\mathrm{e}^{\mathrm{i}t(L_\omega+Q)}$ defines a W^*-dynamics on $\mathcal{O}_\omega$. This extension of perturbation theory has been developed in [DJP].

Except for its important role in Araki's perturbation theory of KMS-states (see Section 3 in [KMS states] and [BR2]), the operator $L_\omega + Q$ is of little value in the study of dynamical properties of τ_V. This is due to the fact that it is not adapted to the structure of the enveloping von Neumann algebra $\mathcal{O}_\omega$. The standard Liouvillean, introduced in [Tomita-Takesaki theory], corrects this problem.

If $t \mapsto V(t) = V(t)^* \in \mathcal{O}$ is continuous then equation $\partial_t \tau_V^{s\to t}(A) = \tau_V^{s\to t}(\delta_{V(t)}(A))$ together with the condition $\tau_V^{s\to s} = \mathrm{Id}$ defines a two parameter family of $*$-automorphisms of $\mathcal{O}$ such that $\tau_V^{s\to t} \circ \tau_V^{t\to r} = \tau_V^{s\to r}$. Perturbation theory can be developed as in the time independent case starting from the integral equation

$$\tau_V^{s\to t}(A) = \tau^{t-s}(A) + \int_s^t \tau_V^{s\to u}\left(\mathrm{i}[V(u), \tau^{t-u}(A)]\right) \mathrm{d}u.$$

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