The C^*-algebra approach

JAN DEREZIŃSKI[1], CLAUDE-ALAIN PILLET[2]

[1] Department of Mathematical Methods in Physics
Warsaw University
Hoża 74, 00-682, Warsaw, Poland
jan.derezinski@fuw.edu.pl

[2]CPT-CNRS, UMR 6207
Université du Sud Toulon-Var
B.P. 20132
83957 La Garde Cedex, France
pillet@univ-tln.fr

1 Why operator algebras?

The quantum mechanical description of a system involving only a finite number of particles or degrees of freedom (a finite system) is given by a Hilbert space $\mathcal{H}$ and a Hamiltonian H, a self-adjoint operator on $\mathcal{H}$. States of the system are described by unit vectors $\psi \in \mathcal{H}$ or more generally statistical mixtures, i.e., density matrices. Physical quantities, or observables, are self-adjoint operators on $\mathcal{H}$. The time evolution of the system is governed by its Hamiltonian. In the Schrödinger picture the state evolves according to the Schrödinger equation $\mathrm{i}\partial_t \psi_t = H\psi_t$ while observables are time-independent. In the equivalent Heisenberg picture the state is time-independent and observables evolve according to the Heisenberg equation $\partial_t A_t = \mathrm{i}[H, A_t]$. As a consequence of the tight relation between Hamiltonian and dynamics, the spectrum $\mathrm{sp}(H)$ of H contains a lot of information about the physics of the system.

From a mathematical perspective the construction of $\mathcal{H}$ and H for a given physical system is a problem of representation theory. In the case of a non-relativistic system of N pointlike particles in Euclidean space $\mathbb{R}^3$, the correspondence between the classical Hamiltonian description and the quantum description is given by canonical quantization. This procedure provides a representation of the positions $Q_1, \ldots, Q_{3N}$ and conjugate canonical momenta $P_1, \ldots, P_{3N}$ by self-adjoint operators satisfying the canonical commutation relations (CCR)

$$\mathrm{i}[P_i, Q_j] = \delta_{ij}, \quad \mathrm{i}[P_i, P_j] = 0, \quad \mathrm{i}[Q_i, Q_j] = 0.$$

Physics imposes other constraints. If the particles have spin then $\mathcal{H}$ has to carry N representations of the Lie algebra of $SU(2)$, the quantum mechanical rotation group. If the particles are indistinguishable then Pauli's principle imposes a definite covariance (as prescribed by the spin-statistic theorem) with respect to the natural action of the symmetric group S_N on $\mathcal{H}$.

The deep fact about finite systems is that the resulting quantum description is unique, up to unitary transformations and mostly irrelevant multiplicity. This is the content of the celebrated Stone-von Neumann theorem (see [Ro] for a review). In particular the spectrum of the Hamiltonian of the system is uniquely determined (again up to some global multiplicity which can usually be reduced by super-selection rules).

When dealing with systems with an infinite number of particles or degrees of freedom – quantum fields or quantum statistical mechanics in the thermodynamic limit – we are faced with a radically different situation. The breakdown of Stone-von Neumann theorem implies the existence of a large number of unitary inequivalent irreducible representations of the CCR. This phenomenon is not a special feature of CCR. The following example shows that it occurs also for representations of the group $SU(2)$ (see also [Th]).

Consider an infinite chain of quantum spins $s = 1/2$. To each site $x \in \mathbb{Z}$ we associate observables $\sigma_x^{(1)}$, $\sigma_x^{(2)}$ and $\sigma_x^{(3)}$ which satisfy the $SU(2)$ commutation relations

$$[\sigma_x^{(j)}, \sigma_y^{(k)}] = 2\mathrm{i}\delta_{xy}\epsilon_{jkl}\sigma_x^{(l)}. \tag{1}$$

The Hamiltonian is formally given by $H = J\sum_{x\in\mathbb{Z}} \sigma_x^{(3)}$ so that

$$\mathrm{i}[H, \sigma_x^{(j)}] = -2J\epsilon_{3jk}\sigma_x^{(k)}. \tag{2}$$

Set $\mathfrak{h}_x = \mathbb{C}^2$ for any $x \in \mathbb{Z}$. The naive candidate for the Hilbert space of the system is the maximal tensor product of all the spaces $\mathfrak{h}_x$, which we will denote $\otimes_{x\in\mathbb{Z}}\mathfrak{h}_x$. It is defined as the completion of the pre-Hilbert space spanned by vectors of the form $\otimes_{x\in\mathbb{Z}}\varphi_x$, where each φ_x is a unit vector in $\mathfrak{h}_x$. The inner product between two such vectors is defined by

$$(\otimes_{x\in\mathbb{Z}}\psi_x \mid \otimes_{x\in\mathbb{Z}}\varphi_x) = \prod_{x\in\mathbb{Z}}(\psi_x \,|\, \varphi_x), \tag{3}$$

whenever the infinite product on the left of (3) is absolutely convergent. Otherwise, the inner product on the right of (3) is set to be zero. The space $\otimes_{x\in\mathbb{Z}}\mathfrak{h}_x$ was first considered by von Neumann in [VN] (he called it the complete direct product of the family $(\mathfrak{h}_x)_{x\in\mathbb{Z}}$). This space is much too big for most applications. In particular, it is not separable, i.e., it does not have a countable orthonormal basis[1].

Let us describe another candidate for the notion of the infinite tensor product, which is more useful in quantum physics. For all $x \in \mathbb{Z}$ fix an orthonormal basis $\{\chi_x^-, \chi_x^+\}$ of $\mathfrak{h}_x$. To each finite subset $X \subset \mathbb{Z}$ associate the vector

$$e_X = \left(\bigotimes_{x\in X}\chi_x^+\right)\bigotimes\left(\bigotimes_{x\in\mathbb{Z}\setminus X}\chi_x^-\right).$$

According to (3) one has $(e_X|e_Y) = \delta_{XY}$. Thus, finite linear combinations of the vectors e_X form a pre-Hilbert space. The Hilbert space $\mathcal{H}$ obtained by completion is separable since $\{e_X \,|\, X \subset \mathbb{Z}, |X| < \infty\}$ is a countable orthonormal basis.

We note that a pair $(\mathfrak{h}, \chi)$, where $\mathfrak{h}$ is a Hilbert space and $\chi \in \mathfrak{h}$ a unit vector, is called a grounded Hilbert space. The above construction is a special case of the tensor product of grounded Hilbert spaces, namely $\mathcal{H} = \otimes_{x\in\mathbb{Z}}(\mathfrak{h}_x, \chi_x^-)$. The interested reader should consult [BSZ] for the general construction.

Remark that the maximal tensor product $\otimes_{x\in\mathbb{Z}}\mathfrak{h}_x$ naturally splits into the direct sum of sectors, where each sector has the form $\otimes_{x\in\mathbb{Z}}(\mathfrak{h}_x, \chi_x^-)$ for a certain sequence of unit vectors $\chi_x^- \in \mathfrak{h}_x$.

If $J > 0$ then the ground state of the chain has all spins pointing down in direction 3. If we interpret $\chi_x^\pm$ as the eigenstate of the spin at x in direction 3 with eigenvalue $\pm 1/2$ then the vector $e_\emptyset$ clearly describes this ground state. Then the vector e_X describes a local excitation of the chain, the spins at $x \in X$ pointing up in direction 3. This immediately leads to the following representation of the commutation relations (1) on $\mathcal{H}$

$$\sigma_x^{(1)+}e_X = e_{X\odot x}, \quad \sigma_x^{(2)+}e_X = \mathrm{i}s_X(x)e_{X\odot x}, \quad \sigma_x^{(3)+}e_X = s_X(x)e_X,$$

where

$$X\odot x = \begin{cases} X\setminus\{x\} & \text{if } x\in X,\\ X\cup\{x\} & \text{if } x\notin X,\end{cases} \qquad s_X(x) = \begin{cases} +1 & \text{if } x\in X,\\ -1 & \text{if } x\notin X.\end{cases}$$

We get a different representation of the commutation relations (1) if we think of $\chi_x^\pm$ as the eigenstate of the spin in direction 3 with eigenvalue $\mp 1/2$, namely

$$\sigma_x^{(1)-} = \sigma_x^{(1)+}, \quad \sigma_x^{(2)-} = -\sigma_x^{(2)+}, \quad \sigma_x^{(3)-} = -\sigma_x^{(3)+}.$$

By construction $\sigma_x^{(3)+}e_\emptyset = -e_\emptyset$ for all $x \in \mathbb{Z}$ but one easily checks that there is no unit vector $\Psi \in \mathcal{H}$ such that $\sigma_x^{(3)-}\Psi = -\Psi$ for all $x \in \mathbb{Z}$. Thus, in the second representation of the system the ground state *does not belong to* $\mathcal{H}$. In particular, there is no unitary operator U on $\mathcal{H}$ such that $U\sigma_x^{(3)-}U^* = \sigma_x^{(3)+}$: The two representations are inequivalent. From the fact that

$$e_Y = \prod_{x\in X\Delta Y}\sigma_x^{(1)\pm}e_X,$$

[1] Except for recent developments in quantum gravity most Hilbert spaces of quantum physics are separable.

it follows that the two representations are also irreducible.

Expressing the formal Hamiltonian H in the two representations leads to

$$He_X = J\sum_{x\in X} \pm e_X + J\sum_{x\notin X} \mp e_X = \left(\pm 2J|X| \mp J\sum_{x\in\mathbb{Z}} 1\right) e_X.$$

Discarding a constant – the infinite energy of the state $e_\emptyset$ – we may thus set

$$H_\pm e_X = \pm 2J|X|e_X,$$

which defines two self-adjoint operators on $\mathcal{H}$. The physical meaning of $H_\pm$ is quite clear: It gives the energy of the system relative to the infinite energy of the state $e_\emptyset$. One easily verifies that

$$\mathrm{i}[H_\pm, \sigma_x^{(j)\pm}] = -2J\epsilon_{3jk}\sigma_x^{(k)\pm},$$

i.e., that the commutation relations (2) are satisfied. Note however that the spectra of the two Hamiltonians are quite different

$$\mathrm{sp}(H_\pm) = \pm 2J\mathbb{N}.$$

Of course this does not come as a surprise since $H_\pm$ measures the energy relative to two distinct reference states: One of them has all spins up while the other has all spins down.

In conclusion, there is no natural way to represent the algebraic structure induced by commutation relations (1), (2) in a Hilbert space. To select such a representation one needs to specify a reference state. In equilibrium statistical mechanics this fact does not lead to difficulties since it is always possible to define thermodynamic quantities (free energy, pressure, ...) as limits of quantities related to finite systems. The situation is different in non-equilibrium statistical mechanics where dynamics plays a much more important role. To give a mathematically precise sense to non-equilibrium steady states for example requires the consideration of infinite systems (see [Nonequilibrium Steady States], [NESS in Quantum Statistical Mechanics]).

In the algebraic approach to quantum mechanics the central object is the algebraic structure – in the above example, Relations (1) and (2). Hilbert spaces and Hamiltonians only appear when this structure gets represented by linear operators. Such representations are induced by the states of the system, usually via the Gelfand-Naimark-Segal construction (see Section 3 below). States with different physical properties (e.g., different particle or energy density) lead to inequivalent representations and hence to different spectral properties of the Hamiltonian describing the dynamics.

The mathematical framework of the algebraic approach to quantum mechanics is the theory of operator algebras. C^*-algebras, von Neumann algebras and W^*-algebras are the most commonly used types of operator algebras in this context. There is a huge literature devoted to the subject. Besides [Operator Algebras], [von Neumann Algebras – General Theory], the reader may consult [D1, D2, KR, S, SZ, T] for mathematical introductions and [BR1, BR2, BSZ, H, R, Si, D] for applications to quantum physics and statistical mechanics.

2 Examples of C^*-algebras

To illustrate the algebraic approach we consider a few systems for which C^*-algebras provide a natural framework (see also [Free Bose and Fermi gases – the algebraic approach]). We shall only be concerned with operator algebras here. We refer to [Quantum Dynamical Systems] for examples of dynamics on these algebras.

2.1 Lattice spin systems

To describe a quantum spin system on the infinite lattice Γ (for example $\Gamma = \mathbb{Z}^d$, with $d \geqslant 1$) let $\mathfrak{h}$ be the finite dimensional Hilbert space of a single spin and associate a copy $\mathfrak{h}_x$ of $\mathfrak{h}$ to each $x \in \Gamma$. For finite subsets $\Lambda \subset \Gamma$ set $\mathfrak{h}_\Lambda \equiv \otimes_{x\in\Lambda}\mathfrak{h}_x$ and define the local C^*-algebras

$$\mathfrak{A}_\Lambda \equiv \mathcal{B}(\mathfrak{h}_\Lambda).$$

If $\Lambda \subset \Lambda'$, the natural isometric injection $A \mapsto A \otimes I_{\mathfrak{h}_{\Lambda' \setminus \Lambda}}$ allows to identify $\mathfrak{A}_\Lambda$ with a subalgebra of $\mathfrak{A}_{\Lambda'}$. With this identification it is possible to define

$$\|A\| = \|A\|_{\mathfrak{A}_\Lambda} \text{ if } A \in \mathfrak{A}_\Lambda,$$

unambiguously for all $A \in \mathfrak{A}_{\mathrm{loc}} \equiv \cup_{\Lambda \subset \Gamma} \mathfrak{A}_\Lambda$, the union being over finite subsets of Γ. This defines a C^*-norm on the $*$-algebra $\mathfrak{A}_{\mathrm{loc}}$. Denote by $\mathfrak{A} = \mathfrak{A}(\Gamma, \mathfrak{h})$ the C^*-algebra obtained as norm completion of $\mathfrak{A}_{\mathrm{loc}}$. We can identify each local algebra $\mathfrak{A}_\Lambda$ with the corresponding subalgebra of $\mathfrak{A}$, hence

$$\mathfrak{A} = \left(\bigcup_{\Lambda \subset \Gamma} \mathfrak{A}_\Lambda \right)^{\mathrm{cl}}, \tag{4}$$

where each $\mathfrak{A}_\Lambda$ is a full matrix algebra. C^*-algebras of this type are called uniformly hyperfinite (UHF) or Glimm-algebras.

2.2 CAR algebras

Let $\mathfrak{h}$ be the Hilbert space of a single fermion (typically $\mathfrak{h} = L^2(\mathbb{R}^d) \otimes \mathbb{C}^n$, but other geometries and additional internal degrees of freedom may lead to different single particle Hilbert spaces). A system of such fermions is described by the second quantization formalism. Denote by $\Gamma_{\mathrm{a}}(\mathfrak{h})$ the fermionic (or anti-symmetric) Fock space over $\mathfrak{h}$. The fermion creation/annihilation operators $a^*(f)$, $a(f)$ are bounded operators on $\Gamma_{\mathrm{a}}(\mathfrak{h})$ satisfying the Canonical Anti-commutation Relations (CAR)

$$[a(f), a^*(g)]_+ = (f|g)I, \quad [a(f), a(g)]_+ = 0, \tag{5}$$

for all $f, g \in \mathfrak{h}$. A convenient choice of observables is the set of polynomials in a, a^*, i.e., the set of finite linear combinations of monomials

$$a^\#(f_1) \cdots a^\#(f_n),$$

where each $a^\#$ stands for either a or a^* and the f_j are elements of $\mathfrak{h}$ or more generally of some subspace $\mathfrak{h}_0 \subset \mathfrak{h}$. This set is clearly a unital $*$-algebra. Its norm closure is a C^*-subalgebra of $\mathcal{B}(\Gamma_{\mathrm{a}}(\mathfrak{h}))$. It turns out that this algebra is completely characterized by $\mathfrak{h}_0$ and the CAR (5).

Theorem 2.1 *Let $\mathfrak{h}_0$ be a pre-Hilbert space. Up to $*$-isomorphisms, there exists a unique unital C^*-algebra $\mathrm{CAR}(\mathfrak{h}_0)$ with the two following properties:*

(i) There exists an anti-linear map $a : \mathfrak{h}_0 \to \mathrm{CAR}(\mathfrak{h}_0)$ such that the CAR (5) hold for any $f, g \in \mathfrak{h}_0$.

(ii) The set of monomials $\{a^\#(f_1) \cdots a^\#(f_n) \,|\, f_1, \ldots, f_n \in \mathfrak{h}_0\}$ is total in the algebra $\mathrm{CAR}(\mathfrak{h}_0)$.

Remarks. 1. For $f \neq 0$, Equ. (5) yields that $a(f) \neq 0$ and that $(a^*(f)a(f))^2 = \|f\|^2 a^*(f)a(f)$. Thus

$$\|a(f)\| = \|f\|, \tag{6}$$

holds for any $f \in \mathfrak{h}_0$. In particular, the map a is continuous, extends to the completion $\mathfrak{h}$ of $\mathfrak{h}_0$ and $\mathrm{CAR}(\mathfrak{h}) = \mathrm{CAR}(\mathfrak{h}_0)$.
2. By condition *(ii)* the algebra $\mathrm{CAR}(\mathfrak{h})$ is separable if and only if the Hilbert space $\mathfrak{h}$ is.
3. An antilinear map $f \mapsto b(f)$ from $\mathfrak{h}$ to $\mathcal{B}(\mathcal{H})$ satisfying (5) extends to a faithful representation of $\mathrm{CAR}(\mathfrak{h})$ in the Hilbert space $\mathcal{H}$.

A direct consequence of this theorem is the following

Corollary 2.2 *Let $\mathfrak{h}_1, \mathfrak{h}_2$ be two Hilbert spaces. Assume that the bounded linear map $U : \mathfrak{h}_1 \to \mathfrak{h}_2$ and the bounded anti-linear map $V : \mathfrak{h}_1 \to \mathfrak{h}_2$ satisfy*

$$\begin{aligned} U^*U + V^*V &= I_{\mathfrak{h}_1}, \qquad & UU^* + VV^* &= I_{\mathfrak{h}_2}, \\ U^*V + V^*U &= 0 \quad, \qquad & UV^* + VU^* &= 0. \end{aligned}$$

Then there exists a unique $*$-isomorphism $\gamma : \mathrm{CAR}(\mathfrak{h}_1) \to \mathrm{CAR}(\mathfrak{h}_2)$ such that

$$\gamma(a(f)) = a(Uf) + a^*(Vf).$$

γ is called the Bogoliubov isomorphism induced by the pair (U, V). If $V = 0$ then U must be unitary. In this case we say that γ is the Bogoliubov isomorphism (or automorphism if $\mathfrak{h}_1 = \mathfrak{h}_2$) induced by U.

When dealing with compound Fermi systems the following result is often useful.

Theorem 2.3 (Exponential law for fermions) *Let $\mathfrak{h}_1, \mathfrak{h}_2$ be two Hilbert spaces. There is a unique unitary operator $U : \Gamma_\mathrm{a}(\mathfrak{h}_1 \oplus \mathfrak{h}_2) \to \Gamma_\mathrm{a}(\mathfrak{h}_1) \otimes \Gamma_\mathrm{a}(\mathfrak{h}_2)$ such that*

$$U\Omega_\mathrm{F} = \Omega_\mathrm{F} \otimes \Omega_\mathrm{F}, \quad Ua(f_1 \oplus f_2)U^* = a(f_1) \otimes I + (-1)^N \otimes a(f_2),$$

where Ω_F denotes the Fock vacuum vector and $N = \mathrm{d}\Gamma(I)$ the number operator.

Note that $N \in \mathrm{CAR}(\mathfrak{h}_1)$ if and only if $\mathfrak{h}_1$ is finite dimensional. Thus, $\mathrm{CAR}(\mathfrak{h}_1 \oplus \mathfrak{h}_2)$ is $*$-isomorphic to the C^*-tensor product $\mathrm{CAR}(\mathfrak{h}_1) \otimes \mathrm{CAR}(\mathfrak{h}_2)$ if and only if at least one of the space $\mathfrak{h}_1$, $\mathfrak{h}_2$ is finite dimensional.

2.3 CCR algebras

Systems of bosons are described in an analogous way by creation/annihilation operators $a^*(f)$ and $a(f)$ on the bosonic (or symmetric) Fock space $\Gamma_\mathrm{s}(\mathfrak{h})$. These operators satisfy the Canonical Commutation Relations

$$[a(f), a^*(g)] = (f|g), \quad [a(f), a(g)] = 0, \tag{7}$$

for $f, g \in \mathfrak{h}$. However, dealing with bosonic systems is more delicate since the operators $a^*(f)$ and $a(f)$ are unbounded. This follows readily from the algebraic structure described by the CCR. Indeed, suppose that $a(f)$ is bounded, then since $a^*(f)a(f)$ is positive it follows from the CCR that $\|a(f)a^*(f)\| = \|a^*(f)a(f)\| + \|f\|^2$ which contradicts the fact that $\|a(f)a^*(f)\| = \|a^*(f)a(f)\| = \|a(f)\|^2$.

Thus it is not a priori clear how to interpret the CCR without referring to some domain $\mathcal{D} \subset \Gamma_\mathrm{s}(\mathfrak{h})$ on which they are supposed to hold. The operator

$$\frac{1}{\sqrt{2}}\left(a^*(f) + a(f)\right),$$

is essentially self-adjoint on the dense subspace $\Gamma_{\mathrm{s,fin}}(\mathfrak{h})$ of finite particle vectors of $\Gamma_\mathrm{s}(\mathfrak{h})$. Its selfadjoint closure is called Segal field operator and denoted by $\phi(f)$. Segal field operators satisfy the commutation relations $[\phi(f), \phi(g)] = \mathrm{i}\,\mathrm{Im}(f|g)$ which are formally equivalent to (7). The unitary operators

$$W(f) = \mathrm{e}^{\mathrm{i}\phi(f)},$$

are called Weyl operators. They satisfy the Weyl relations

$$W(f)W(g) = \mathrm{e}^{-\mathrm{i}\,\mathrm{Im}(f,g)/2}W(f+g). \tag{8}$$

Finite linear combinations of Weyl operators build a $*$-algebra. Its closure is a C^*-algebra which is completely characterized by the Weyl relations (8).

Theorem 2.4 *Let $\mathfrak{h}_0$ be a pre-Hilbert space. Up to $*$-isomorphisms, there exists a unique unital C^*-algebra $\mathrm{CCR}(\mathfrak{h}_0)$ with the following properties:*

(i) There is a map $f \mapsto W(f)$ *from* $\mathfrak{h}_0$ *to* $\mathrm{CCR}(\mathfrak{h}_0)$ *such that*

$$W(-f) = W(f)^*, \quad W(0) \neq 0,$$

and the Weyl relations (8) are satisfied for all $f, g \in \mathfrak{h}_0$.

(ii) The set $\{W(f) \mid f \in \mathfrak{h}_0\}$ *is total in* $\mathrm{CCR}(\mathfrak{h}_0)$.

Remarks. 1. It follows from (8) and conditions (i)-(ii) that $W(0) = I$ and that $W(f)^* = W(f)^{-1}$, i.e., that $W(f)$ is unitary. Moreover, if $f \neq g$ then $\|W(f) - W(g)\| = 2$.
2. Unlike in the CAR-case, if $\mathfrak{h}_0 \neq \mathfrak{h}_1$ then $\mathrm{CCR}(\mathfrak{h}_0) \neq \mathrm{CCR}(\mathfrak{h}_1)$. Moreover, $\mathrm{CCR}(\mathfrak{h}_0)$ is not separable if $\mathfrak{h}_0 \neq \{0\}$.
3. A map $f \mapsto W_\pi(f)$ from $\mathfrak{h}_0$ to the unitary operators on $\mathcal{H}$ satisfying the Weyl relations (8) extends to a representation $(\mathcal{H}, \pi)$ of $\mathrm{CCR}(\mathfrak{h}_0)$.

Bogoliubov isomorphisms between CCR algebras are defined in a similar way than in the CAR case.

Corollary 2.5 *Let* $\mathfrak{h}_1$, $\mathfrak{h}_2$ *be two pre-Hilbert spaces and* $U : \mathfrak{h}_1 \to \mathfrak{h}_2$ *an invertible real-linear map such that* $\mathrm{Im}(Uf|Ug) = \mathrm{Im}(f|g)$. *Then there is a unique* $*$*-isomorphism* $\gamma : \mathrm{CCR}(\mathfrak{h}_1) \to \mathrm{CCR}(\mathfrak{h}_2)$ *such that* $\gamma(W(f)) = W(Uf)$.

An exponential law similar to Theorem 2.3 holds for bosons.

Theorem 2.6 (Exponential law for bosons) *Let* $\mathfrak{h}_1$, $\mathfrak{h}_2$ *be two Hilbert spaces. There is a unique unitary operator* $U : \Gamma_{\mathrm{s}}(\mathfrak{h}_1 \oplus \mathfrak{h}_2) \to \Gamma_{\mathrm{s}}(\mathfrak{h}_1) \otimes \Gamma_{\mathrm{s}}(\mathfrak{h}_2)$ *such that*

$$U\Omega_{\mathrm{F}} = \Omega_{\mathrm{F}} \otimes \Omega_{\mathrm{F}}, \quad UW(f_1 \oplus f_2)U^* = W(f_1) \otimes I + I \otimes W(f_2),$$

where Ω_{F} *denotes the Fock vacuum vector. Thus,* $\mathrm{CCR}(\mathfrak{h}_1 \oplus \mathfrak{h}_2)$ *is* $*$*-isomorphic to the* *C^*-tensor product* $\mathrm{CCR}(\mathfrak{h}_1) \otimes \mathrm{CCR}(\mathfrak{h}_2)$.

In practice the C^*-algebra $\mathrm{CCR}(\mathfrak{h}_0)$ of Theorem 2.4 is not very convenient and one often prefers to work with von Neumann algebras when dealing with bosons. Let $(\mathcal{H}, \pi)$ be a representation of $\mathrm{CCR}(\mathfrak{h}_0)$. The von Neumann algebra on $\mathcal{H}$ generated by $\{\pi(A) \mid A \in \mathrm{CCR}(\mathfrak{h}_0)\}$ is given by the bicommutant

$$\mathfrak{M}_\pi(\mathfrak{h}_0) = \pi(\mathrm{CCR}(\mathfrak{h}_0))''.$$

It is the enveloping von Neumann algebra of the representation π (see [Free Bose and Fermi gases – the algebraic approach] for an example).

2.4 Quasi-local structure of $\mathfrak{A}(\Gamma, \mathfrak{h})$ and $\mathrm{CAR}(\mathfrak{h})$

As already mentioned, in most physical applications the single fermion Hilbert space is $\mathfrak{h} = L^2(\mathbb{R}^d) \otimes \mathbb{C}^n$ or some straightforward variant of it. We can assign to each bounded open subset $\Lambda \subset \mathbb{R}^d$ a local Hilbert space $\mathfrak{h}_\Lambda = L^2(\Lambda) \otimes \mathbb{C}^n$. The canonical isometric injections $\mathfrak{h}_\Lambda \hookrightarrow \mathfrak{h}$ yield injections $\mathrm{CAR}(\mathfrak{h}_\Lambda) \hookrightarrow \mathrm{CAR}(\mathfrak{h})$ which allow us to identify the local algebra $\mathrm{CAR}(\mathfrak{h}_\Lambda)$ with a C^*-subalgebra of $\mathrm{CAR}(\mathfrak{h})$. It follows immediately from remark 1 that

$$\mathrm{CAR}(\mathfrak{h}) = \left(\bigcup_{\Lambda \subset \mathbb{R}^d} \mathrm{CAR}(\mathfrak{h}_\Lambda) \right)^{\mathrm{cl}},$$

which should be compared with Equ. (4). Note however that while $\Lambda \cap \Lambda' = \emptyset$ implies $[\mathfrak{A}_\Lambda, \mathfrak{A}_{\Lambda'}] = \{0\}$, the CAR algebras of disjoint subsets do not commute. This is of course due to the fact that $a(f)$ and $a(g)$ rather anticommute. Let θ be the $*$-automorphism defined by $\theta(a(f)) = -a(f)$ and denote by

$$\mathrm{CAR}_\pm(\mathfrak{h}) = \{A \in \mathrm{CAR}(\mathfrak{h}) \,|\, \theta(A) = \pm A\},$$

the even and odd parts of $\mathrm{CAR}(\mathfrak{h})$ with respect to θ. Alternatively, these are the closed linear spans of monomials of even and odd degrees in the $a^\#$. Then one has $\mathrm{CAR}(\mathfrak{h}_\Lambda) = \mathrm{CAR}_+(\mathfrak{h}_\Lambda) \oplus \mathrm{CAR}_-(\mathfrak{h}_\Lambda)$ and one easily checks that

$$[\mathrm{CAR}_\pm(\mathfrak{h}_\Lambda), \mathrm{CAR}_\pm(\mathfrak{h}_{\Lambda'})] = \{0\}, \quad [\mathrm{CAR}_\pm(\mathfrak{h}_\Lambda), \mathrm{CAR}_\mp(\mathfrak{h}_{\Lambda'})]_+ = \{0\}.$$

From a physical point of view observables localized in disjoint regions of space should be simultaneously measurable. Hence physical observables of a fermionic system should be elements of the even subalgebra $\mathrm{CAR}_+(\mathfrak{h})$. In fact, the stronger requirement of gauge-invariance further reduces the observable algebra to the subalgebra of $\mathrm{CAR}_+(\mathfrak{h})$ generated by monomials in the $a^\#$ containing the same number of a and a^* (see [Araki-Wyss Representation] for a discussion of this point). In both the UHF-algebra $\mathfrak{A}$ and the CAR-algebra $\mathrm{CAR}(\mathfrak{h})$, the local subalgebras define a so called quasi-local structure. We refer to Section 2.6 of [BR1] for a general discussion.

3 States and the GNS construction

Let $\mathcal{O}$ be a C^*-algebra. A linear functional φ on $\mathcal{O}$ is positive if $\varphi(A^*A) \geq 0$ for all $A \in \mathcal{O}$. A positive linear functional is automatically bounded, i.e., an element of the dual $\mathcal{O}^\#$. It is called a state if $\|\varphi\| = 1$. If $\mathcal{O}$ has a unit I then $\|\varphi\| = \varphi(I)$ for any positive linear functional φ.

A representation of $\mathcal{O}$ on a Hilbert space $\mathcal{H}$ is a $*$-morphism $\pi : \mathcal{O} \to \mathcal{B}(\mathcal{H})$. Given such a representation and a unit vector $\Omega \in \mathcal{H}$, the formula $\varphi(A) = (\Omega|\pi(A)\Omega)$ defines a state on $\mathcal{O}$. The GNS construction shows that any state on $\mathcal{O}$ is of this form.

Theorem 3.1 *Let ω be a state on the C^*-algebra $\mathcal{O}$. Then there exist a Hilbert space $\mathcal{H}_\omega$, a representation π_ω of $\mathcal{O}$ in $\mathcal{H}_\omega$ and a unit vector $\Omega_\omega \in \mathcal{H}$ such that*

1. *$\omega(A) = (\Omega_\omega|\pi_\omega(A)\Omega_\omega)$ for all $A \in \mathcal{O}$.*
2. *$\pi_\omega(\mathcal{O})\Omega_\omega$ is dense in $\mathcal{H}_\omega$.*

The triple $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ is unique up to unitary equivalence. It is called the GNS representation or the cyclic representation of $\mathcal{O}$ induced by the state ω.

An important object associated with the GNS representation is the enveloping von Neumann algebra: The σ-weak closure $\mathcal{O}_\omega$ of $\pi_\omega(\mathcal{O})$ in $\mathcal{B}(\mathcal{H}_\omega)$. By von Neumann's bicommutant theorem, it is given by the bicommutant

$$\mathcal{O}_\omega = \pi_\omega(\mathcal{O})''.$$

We note that if $\mathcal{O}$ is itself a von Neumann or a W^*-algebra and ω is σ-weakly continuous (i.e., a normal state) then π_ω is σ-weakly continuous and $\mathcal{O}_\omega = \pi_\omega(\mathcal{O})$.

Two states ω, ν on $\mathcal{O}$ are quasi-equivalent if there exists a $*$-isomorphism $\phi : \mathcal{O}_\omega \to \mathcal{O}_\nu$ such that $\pi_\nu = \phi \circ \pi_\omega$.

The folium of a state ω is the set $\mathcal{N}_\omega$ of all states of the form $\nu(A) = \mathrm{tr}\rho\pi_\omega(A)$ for some density matrix ρ on $\mathcal{H}_\omega$. A state $\nu \in \mathcal{N}_\omega$ is said to be ω-normal. Thus, ω-normal states on $\mathcal{O}$ are characterized by the fact that they extend to normal states on the enveloping von Neumann algebra $\mathcal{O}_\omega$.

Theorem 3.2 *The following propositions are equivalent.*

1. *μ and ν are quasi-equivalent.*

2. *μ and ν have the same folium.*

3. *There exists Hilbert spaces $\mathcal{K}_\mu$ and $\mathcal{K}_\nu$ and a unitary map $V : \mathcal{H}_\mu \otimes \mathcal{K}_\mu \to \mathcal{H}_\nu \otimes \mathcal{K}_\nu$ such that*

$$\pi_\nu \otimes I = V(\pi_\mu \otimes I)V^*.$$

The reader should consult [BR1] for a more detailed discussion.

References

[BR1] Bratteli, O., Robinson D. W.: *Operator Algebras and Quantum Statistical Mechanics 1.* Second edition, Springer, Berlin (2002).

[BR2] Bratteli, O., Robinson D. W.: *Operator Algebras and Quantum Statistical Mechanics 2.* Second edition, Springer, Berlin (2002).

[BSZ] Baez, J.C., Segal, I.E., Zhou, Z.: *Introduction to Algebraic and Constructive Quantum Field Theory.* Princeton University Press, Princeton, N.J. (1992).

[D] Derezínski, J.: Introduction to representations of canonical commutation and anticommutation relations. In J. Derezínski and H. Siedentop, editors, *Large Coulomb Systems - QED.* Lecture Notes in Physics **695**, 145. Springer, New York, (2006).

[D1] Dixmier, J.: *C^*-Algebras.* North-Holland, Amsterdam (1977).

[D2] Dixmier, J.: *Von Neumann Algebras.* North-Holland, Amsterdam (1981).

[H] Haag, R.: *Local Quantum Physics: Fields, Particles, Algebras.* Springer, Berlin (1992).

[KR] Kadison, R.V., Ringrose, J.R.: *Fundamentals of the Theory of Operator Algebras I.* AMS, Providence, R.I. (1997).

[R] Ruelle, D.: *Statistical Mechanics. Rigorous Results.* Benjamin, New York (1969).

[Ro] Rosenberg, J.: A selective history of the Stone–von Neumann theorem. Contemp. Math. **365**, 123 (2004).

[S] Sakai, S.: *C^*-Algebras and W^*-Algebras.* Springer, Berlin (1998).

[Si] Simon, B.: *The Statistical Mechanics of Lattice Gases.* Princeton University Press, Princeton, N.J. (1993).

[SZ] Stratila, S., Zsidó, L.: *Lectures on von Neumann Algebras.* Editura Academiei, Bucharest (1979).

[T] Takesaki, M.: *Theory of Operator Algebras I.* Springer, Berlin (2002).

[Th] Thirring, W.: Quantum Mechanics of Large Systems. Springer, New York (1983).

[VN] Von Neumann, J.: On infinite direct products. Comp. Math. **6**, 1 (1939).