# Repeated Quantum Interactions and Discrete-Time Quantum Noises.

S. Attal

I. Completely Positive Maps

II. Repeated Quantum Interactions

III. Probabilistic Interpretations

IV. A First Approach to the Continuous-Time

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$$\mathcal{L}(\rho) = \mathrm{tr}_{\mathcal{H}_E}\left(U(\rho \otimes \omega)U^*\right) \quad \text{for states}$$

or

$$\mathcal{L}^*(X) = \mathrm{tr}_\omega\left(U^*(X \otimes I)\,U\right) \quad \text{for observables.}$$

$$\boxed{\mathcal{L}(\rho) = \sum_i \Pi_i \, \rho \, \Pi_i^*}$$

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<u>Krauss representation</u>

$\mathcal{L}$ and $\mathcal{L}^*$ are <u>Completely Positive Maps</u>

$\mathcal{L}$ is a <u>quantum channel</u> (it preserves the trace)

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$\mathcal{L}$ = most general transform of $\mathcal{H}_S$

$\rho_n(\rho) = \mathcal{L}^n(\rho)$ , $n \in \mathbb{N}$.

or $\rho_{nh}(\rho) = \mathcal{L}^n(\rho)$ , $n \in \mathbb{N}$.

$$\begin{tikzcd}
\mathcal{L}_1(\mathcal{H}_S \otimes \mathcal{H}_E) \arrow[r, "U \cdot U^*"] & \mathcal{L}_1(\mathcal{H}_S \otimes \mathcal{H}_E) \arrow[d, "\mathrm{tr}_{\mathcal{H}_E}"] \\
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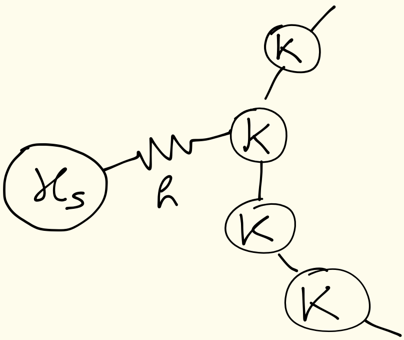

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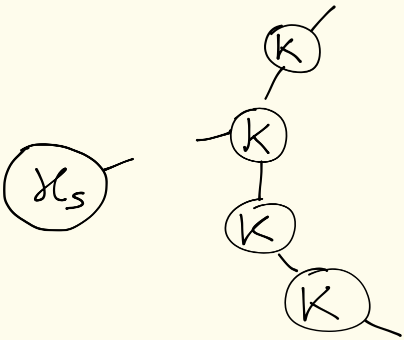

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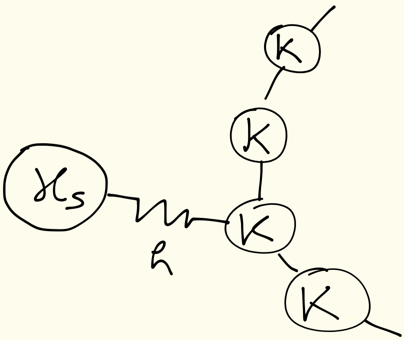

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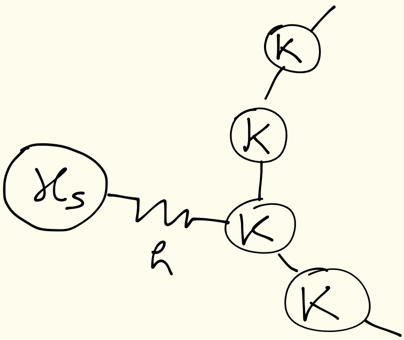


On $\mathcal{H}_S \otimes K$: $\quad U = e^{-ih H_{tot}}$

$$\boxed{U = \sum_{i,j} U^i_j \otimes a^i_j}$$

$U^i_j \in \mathcal{B}(\mathcal{H}_S)$, $a^i_j e_k = \delta_{ik} e_j$.

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$$\begin{cases} V_{n+1} = U_{n+1} V_n \\ V_0 = I \end{cases}$$

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$$V_{n+1} = \sum_{i,j} U_j^i \otimes a_j^i(n+1)\; V_n$$

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$$\boxed{V_{n+1} - V_n = \sum_{i,j} \left(U^i_j - \delta_{ij} I\right) V_n \, a^i_j(n+1)}$$

## Theorem

For all $n \in \mathbb{N}$:

$$\mathrm{tr}_{\mathcal{H}_E}\left( V_n \, (\rho \otimes \omega) \, V_n^* \right) = \mathcal{L}^n(\rho) .$$

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$X, Y$ etc... random variables on $(\Omega, \mathcal{F}, \mathbb{P})$.

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How can we read the probabilistic properties of $X, Y$ on $\mathcal{H}$? Law, independence, functional calculus, ... ?

Certainly not throught $x = UX$, $y = UY$, ... which can be any element of $\mathcal{H}$.

Good object:

$$\mathcal{M}_X : \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$$
$$Z \longmapsto XZ$$

self-adjoint operator.

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$$Q = U \mathcal{M}_X U^*$$
$$\varphi = U \mathbb{1}$$

$$f(Q) = U \mathcal{M}_{f(X)} U^*$$

$$\langle \varphi | f(Q) | \varphi \rangle = \mathbb{E}[f(X)]$$

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Introducing $Y$: $R = U \mathcal{M}_Y U^*$ commutes with $Q$

$$\langle \varphi | g(Q, R) | \varphi \rangle = \mathbb{E}[g(X, Y)]$$

$\Rightarrow$ joint law, independence, ...

Independant random variables $(\nu_n)_{n\in\mathbb{N}}$

$$p\,\delta_1 + q\,\delta_0 ,$$

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$$X_n = \frac{\vartheta_n - p}{\sqrt{pq}}$$

$$\mathbb{E}[X_n] = 0,\quad \mathbb{E}[X_n^2] = 1$$

$$X_n = \begin{cases} \sqrt{q/p} & \text{with prob } p \\ -\sqrt{p/q} & \text{with prob } q \end{cases}$$

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1) On $L^2(\Omega, \mathcal{F}, \mathbb{P})$, the products

$$X_{i_1} \cdots X_{i_k}$$

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2) $$\boxed{\mathcal{M}_{X_n} = a^0_1(n) + a^1_0(n) + c_p\, a^1_1(n)}$$

$c_p = \dfrac{q-p}{\sqrt{pq}}$ bij $(0,1)\longrightarrow \mathbb{R}$.

# Example 1

$p = 1/2$, $X_n = \pm 1$ prob $1/2$

$c_p = 0$

$\mathcal{B}_{X_n} = a_1^0(n) + a_0^1(n) \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$

# Example 1

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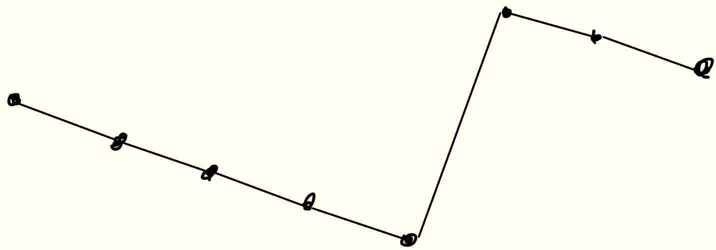
# Example 2

$p = h, \quad X_n = \begin{cases} 1/\sqrt{h} & \text{prob } h \\ -\sqrt{h} & \text{prob } 1-h \end{cases}$



$c_p \sim \frac{1}{\sqrt{h}}$

$\mathcal{A}_{X_n} = a_1^0(n) + a_0^1(n) + \frac{1}{\sqrt{h}} a_1^1(n).$

In particular, among the quantum dynamics

$$V_{n+1} = \sum_{i,j} U^i_j V_n a^i_j(n+1),$$

some are purely quantum:

$$V_{n+1} = K V_n \otimes I_{n+1} + A V_n a^0_1(n+1) + A^* V_n a^1_0(n+1),$$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{(spontaneous emission)}$$

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and some are driven by classical noises:

$$V_{n+1} = K V_n \otimes I_{n+1} + L V_n \otimes \left(a_1^0(n+1) + a_0^1(n+1)\right)$$

$$\boxed{V_{n+1} = K V_n \otimes I_{n+1} + L V_n X_{n+1}}$$

Random walk on $\mathcal{U}(\mathcal{H}_s)$ driven by symmetric Bernoulli.

Conclusion: in the eq.

$$V_{n+1} = \sum_{i,j} U^i_j \, V_n \, a^i_j(n+1)$$

the $a^i_j(n)$ are the quantum noises, they encode all the possible actions of the environment. They contain all the classical noises (random walks) as particular cases.

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## IV An approach to the continuous-time limit.

On $\mathcal{H}_S \otimes \bigotimes_{h\mathbb{N}} K$ we have the evolution

$$\boxed{V_{(n+1)h} - V_{nh} = \sum_{i,j} \left(U^i_j - \delta_{ij} I\right) V_{nh} \, a^i_j((n+1)h)}$$

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In the continuous-time limit we should end up with a state space $\mathcal{H} \otimes \bigotimes_{\mathbb{R}^+} K$ (which sense?) and an evolution

$$\boxed{dV_t = \sum_{i,j} L^i_j \, V_t \, da^i_j(t)}$$

(which $da^i_j(t)$? They should contain B.M. and P.P.)

$$F = \sum_{i_1, \dots, i_k} f(i_1, \dots, i_k) \; e_{i_1} \otimes \cdots \otimes e_{i_k}$$

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For convergence: $f(ih) = \sqrt{h} \, g(ih)$ with $g \in L^2(\mathbb{R}^+)$.

$$F = \sum_i g(ih) \, dX_{ih}$$

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$$\|F\|^2 = \sum_i |g(ih)|^2 h \longrightarrow \int_{\mathbb{R}^+} |g(s)|^2 \, ds .$$

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$$\bigotimes_{n \in h\mathbb{N}} \mathbb{C}^k \subset \bigotimes_{n \in \frac{h}{2}\mathbb{N}} \mathbb{C}^k \subset \cdots$$

In that case, the only way to make the operators $a_j^i(nh)$ acting non-trivially in the limit is by taking:

$$da_0^0(nh) = h\, a_0^0(nh)$$

$$da_1^0(nh) = \sqrt{h}\, a_1^0(nh)$$

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The table

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|---|---|---|---|---|
| $a_0^0$ | $a_0^0$ | $0$ | $a_0^1$ | $0$ |
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We also expect that

$$da^0_1(t) + da^1_0(t) = dW_t \quad \text{(Brownian motion)}$$

$$da^0_1(t) + da^1_0(t) + da^1_1(t) = dN_t - dt \quad \text{(Compensated Poisson process)}$$

becomes

| $\curvearrowright$ | $da_0^0$ | $da_1^0$ | $da_0^1$ | $da_1^1$ |
|---|---|---|---|---|
| $da_0^0$ | $0$ | $0$ | $0$ | $0$ |
| $da_1^0$ | $0$ | $0$ | $0$ | $0$ |
| $da_0^1$ | $0$ | $da_0^0$ | $0$ | $da_0^1$ |
| $da_1^1$ | $0$ | $da_1^0$ | $0$ | $da_1^1$ |

We also expect that

$$da_1^0(t) + da_0^1(t) = dW_t \quad \text{(Brownian motion)}$$

$$da_1^0(t) + da_0^1(t) + da_1^1(t) = dN_t - dt \quad \text{(Compensated Poisson process)}$$

compatible with the table: $(da_0^0(t) = dt)$

$$(dW_t)^2 = dt$$

$$(dX_t)^2 = dt + dX_t \qquad (X_t = N_t - t)$$