QUANTUM NOISES
Q.D.S.

S. Attal

I. Quant. Dyn. Semigroups

II. The Probabilistic Fock Space

III Quantum Noises

IV Recovering Classical Noises

V Quantum Langevin Equations

I Quantum Dynamical Semigroups

$\mathcal{P}_{nh}(\rho) = \mathcal{L}_h^n(\rho)$ discrete semigroup of C.P. maps

$$\mathcal{L}_h(\rho) = \sum_i \Pi_i(h)\,\rho\,\Pi_i(h)^*$$

with $\sum_i \Pi_i(h)^* \Pi_i(h) = I$.

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$$\Rightarrow \begin{cases} \Pi_0(h) = I + h L_0 + o(h) \\ \Pi_i(h) = \sqrt{h} L_i + o(\sqrt{h}) \end{cases}$$

The condition $\sum_i \Pi_i(h)^* \Pi_i(h) = I$ then becomes

$$L_0 = iH - \frac{1}{2} \sum_k L_k^* L_k . \qquad (H \text{ self-adjoint})$$

We get

$$\boxed{\mathcal{L}(\rho) = i[H,\rho] - \frac{1}{2}\sum_k \left(L_k^* L_k \rho + \rho L_k^* L_k - 2 L_k \rho L_k^*\right)}$$

Lindblad generator.

Typical generator of $(P_t)_{t\in\mathbb{R}^+}$ semigroup of quantum channels.

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For observables:

$$\boxed{\mathcal{L}^*(X) = -i[H,X] - \frac{1}{2}\sum_k \left(L_k^* L_k X + X L_k^* L_k - 2 L_k^* X L_k\right).}$$

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$$\boxed{\chi_t = a^*(\mathbb{1}_{[0,t]})\,\Omega}$$

$$\Phi_{[t_1,t_2]} = \Gamma_s\left(L^2([t_1,t_2])\right) \subset \Phi$$

$$\Phi \sim \Phi_{[0,t_1]} \otimes \Phi_{[t_1,t_2]} \otimes \cdots \otimes \Phi_{[t_n,+\infty[}.$$

$$\boxed{(\chi_t)_{t\in\mathbb{R}^+} \text{ is unique s.t. } \chi_t - \chi_s \in \Phi_{[s,t]} \quad \forall\, s<t.}$$

The products

$$(X_{t_1+h_1} - X_{t_1}) \otimes \cdots \otimes (X_{t_n+h_n} - X_{t_n})$$

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$$F = c\Omega + \sum_{n=1}^{\infty} \int_{0<t_1<\cdots<t_n} f(t_1, \ldots, t_n)\, dX_{t_1} \otimes \cdots \otimes dX_{t_n}.$$

$$\|F\|^2 = |c|^2 + \sum_{n=1}^{\infty} \int_{0<t_1<\cdots<t_n} |f(t_1, \ldots, t_n)|^2\, dt_1 \cdots dt_n.$$

$dX_\sigma = dX_{t_1} \otimes \cdots \otimes dX_{t_n}$ form a continuous basis of Φ.

$$\Phi = \bigotimes_{t \in \mathbb{R}^+} \mathbb{C}^2_{(t)}$$

where $\mathbb{C}^2_{(t)}$ is generated by Ω_t and dX_t.

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$$\int_0^t H_s \, dA_s$$

operators on Φ

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$$\sum_i H_{t_i} (A_{t_{i+1}} - A_{t_i})$$

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$\to$ composition $\Rightarrow$ domain problems

$\to$ non commution $\quad H_t \, dA_t \neq dA_t \, H_t$

First approach:

$$H_t = H_t \otimes I \quad \text{on} \quad \Phi_{[0,t]} \otimes \Phi_{[t,+\infty[} \qquad \text{(easy)}$$

$$A_t - A_s = I \otimes (A_t - A_s) \otimes I \quad \text{on} \quad \Phi_{[0,s]} \otimes \Phi_{[s,t]} \otimes \Phi_{[t,+\infty[}$$

$$H_{t_i}(A_{t_{i+1}} - A_{t_i}) = H_{t_i} \otimes (A_{t_{i+1}} - A_{t_i})$$

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	Ω	dX_t
$da_0^0(t)$	$dt\,\Omega$	0
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$da_1^1(t)$	0	dX_t

$$a_j^i(t) = \int_0^t da_j^i(s)$$

$$a_j^i(t) f = \int_0^t da_j^i(s) \int_\sigma f(\sigma)\, dX_\sigma$$

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$\Longrightarrow$ explicit formulas, defining concrete operators

$$[a_0^0(t) f](\sigma) = \int_0^t f(\sigma)\, ds = t f(\sigma) = [t f](\sigma)$$

$$[a_1^0(t) f](\sigma) = \sum_{\substack{s \in \sigma \\ s \leq t}} f(\sigma \setminus s)$$

$$[a_0^1(t) f](\sigma) = \int_0^t f(\sigma \cup \{s\})\, ds$$

$$[a_1^1(t) f](\sigma) = \#(\sigma \cap [0,t])\, f(\sigma).$$

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$$S_t = \int_0^t K_s \, da_\ell^k(s)$$

$$T_t S_t = \int_0^t T_s \, dS_s + \int_0^t dT_s \, S_s + \int_0^t dT_s \, dS_s$$

$$= \int_0^t T_s K_s \, da_\ell^k(s) + \int_0^t H_s S_s \, da_j^i(s)$$

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where
$$da_0^1 \, da_1^0 = da_0^0$$
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all other products $= 0$.

Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$, (W_t).

$$L^2(\Omega) \ni F = \mathbb{E}[F] + \sum_{n=1}^{\infty} \int_{0<t_1<\cdots<t_n} f(t_1, \dots, t_n)\, dW_{t_1} \cdots dW_{t_n}$$

$$\|F\|^2 = |\mathbb{E}[F]|^2 + \sum_{n=1}^{\infty} \int_{0<t_1<\cdots<t_n} |f(t_1, \dots, t_n)|^2\, dt_1 \cdots dt_n$$

$$L^2(\Omega) \sim \Phi$$

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Same $X_t = N_t - t$

$$\boxed{\mathcal{J}_{X_t} = a_1^0(t) + a_0^1(t) + a_1^1(t).}$$

On $\mathcal{H}_S \otimes \Phi$

$$dV_t = \sum_{i,j} L_j^i V_t \, da_j^i(t)$$

$L_j^i \in \mathcal{B}(\Phi)$.

$$dV_t = L_0^0 V_t \, dt + \sum_{(i,j) \neq (0,0)} L_j^i V_t \, da_j^i(t).$$

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$$dV_t = \sum_{i,j} L^i_j V_t \, da^i_j(t)$$

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Thm

V_t unitary iff

- $L^1_1 = S - I$, S unitary
- $L^1_0 = -L^{0*}_1 S$
- $L^0_0 = iH - \frac{1}{2} L^{0*}_1 L^0_1$.

Thm

$$\mathrm{tr}_{\Phi}\left[V_t\left(\rho \otimes |\Omega\rangle\langle\Omega|\right) V_t^*\right] = \mathcal{P}_t(\rho)$$

$$\mathcal{P}_t = e^{tL}$$

$$L(\rho) = -i[H,\rho] - \frac{1}{2}\left(L_1^{0*}L_1^0\rho + \rho L_1^{0*}L_1^0 - 2L_1^0\rho L_1^{0*}\right)$$

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$$\mathcal{P}_t=e^{t\mathcal{L}}$$

$$\mathcal{L}(\rho)=-i[H,\rho]-\frac{1}{2}\left(L_1^{0*}L_1^0\rho+\rho L_1^{0*}L_1^0-2L_1^0\rho L_1^{0*}\right)$$

$$dV_t=\left(iH-\frac{1}{2}L^2\right)V_t\,dt+LV_t\,dW_t\qquad(L\text{ s.a.})$$