Open quantum random walks: bi-stability and ballistic diffusion
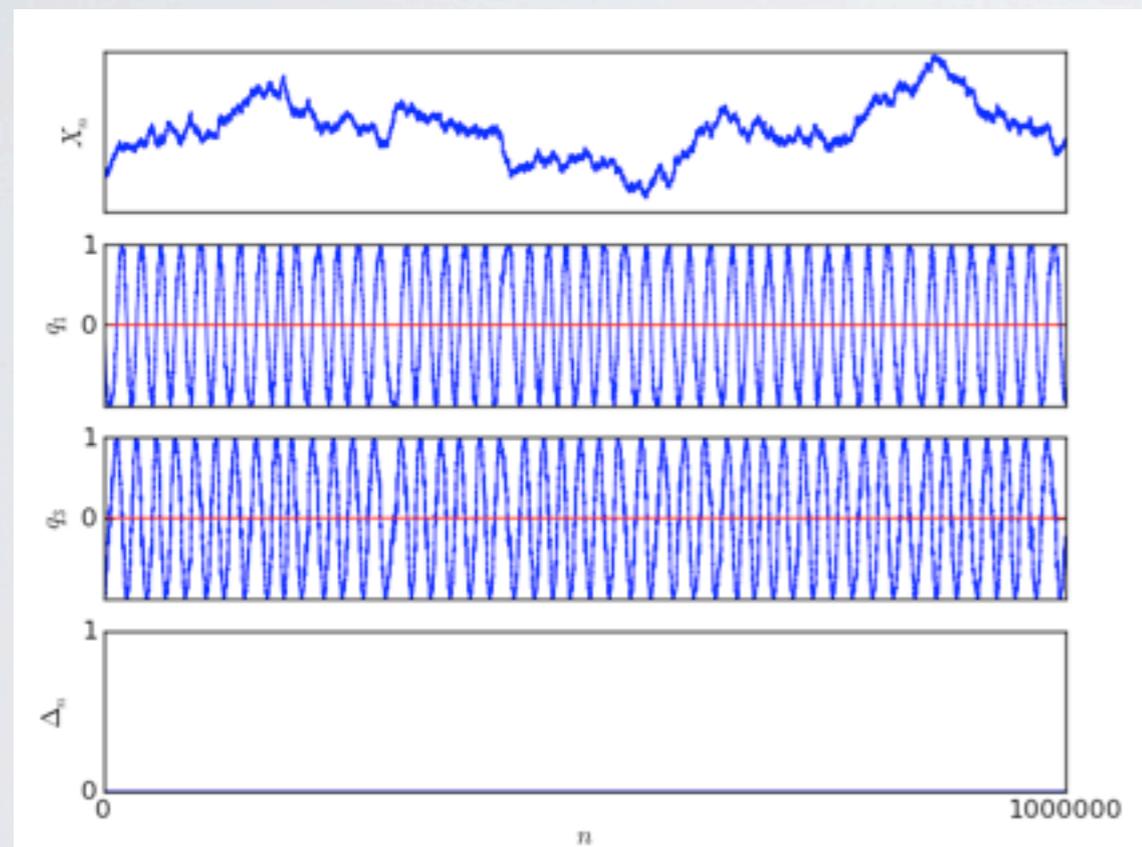
Open quantum brownian motion

with Michel Bauer and Antoine Tilloy
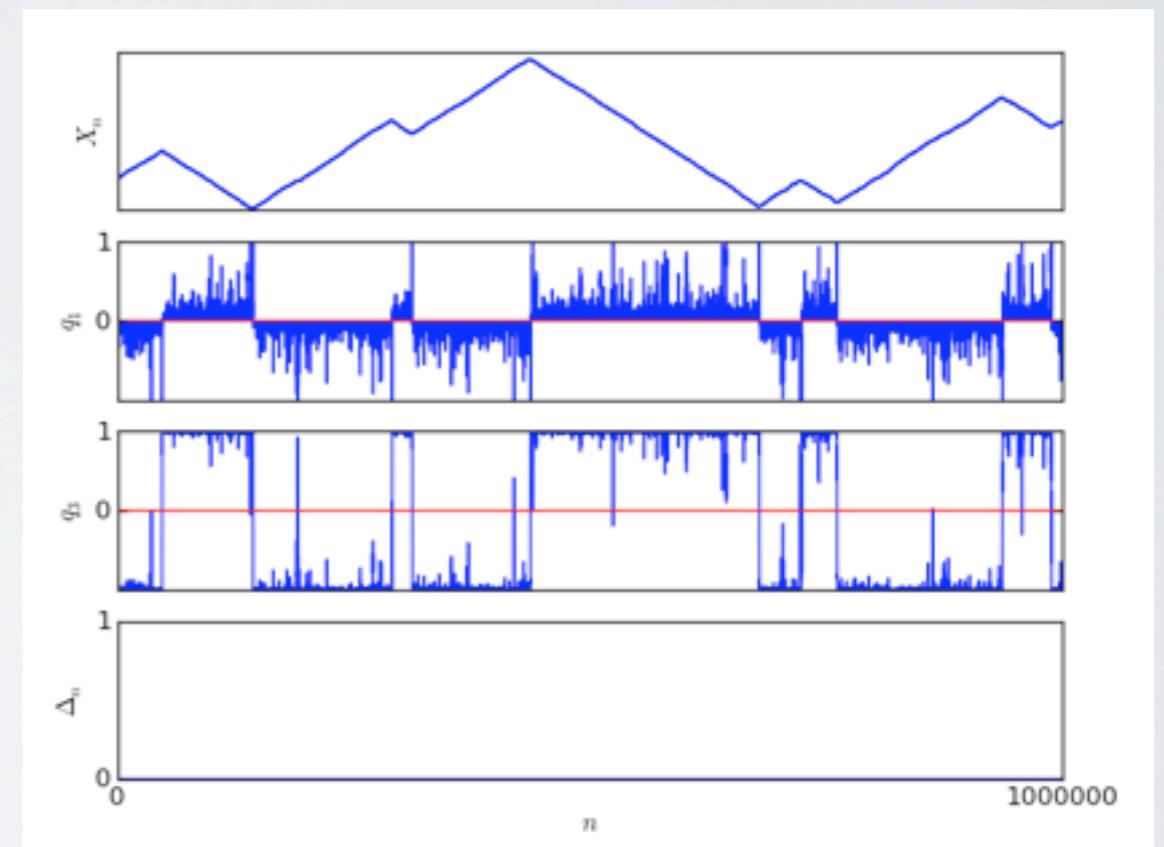
Autrans, July 2013

Different regimes in «open quantum random walks»:

Open quantum random walks describe quantum random motions on the line for systems with internal and orbital degrees of freedom (alias a gyroscope).

A Brownian like regime.

A ballistic like (but diffusive) regime.

On our way, we shall visit the «open quantum Brownian motion»

Outline:

    --- What quantum random walks could be?
      (quantum noise, what objects?)
    --- Open quantum random walk trajectories.
      (a random walk but with an internal quantum gyroscope)
    --- Projection on pure states
      (via sub-martingale theorem, and true more generally)
    --- Two regimes and bi-stability (diffusion or ballistics).
      (gyroscope flips a la Kramer's)

More on...:

    --- Open quantum brownian motion.
    --- Continuous limit
    --- Bi-stability and Q-jumps.

What quantum noise and quantum random walk could be?... or not be!

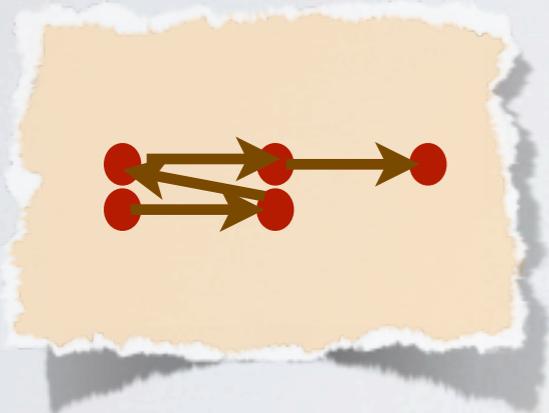*** Classical random walks or stochastic processes.**
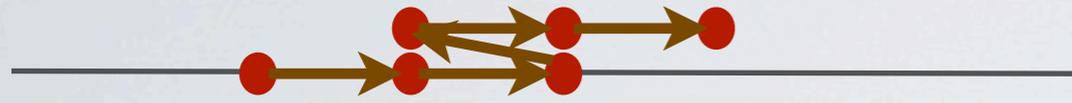
Events: $\quad \omega = (+, +, -, +, \cdots) = (\epsilon_1, \epsilon_2, \epsilon_2, \cdots)$

-- Coins are reset at each time step (i.i.d. variables)

* A quantum Brownian motion (via open system models)

Feymann-Vernon, Keldish, Caldeira-Leggett, Froelich,....

--- A free (quantum) particle in contact with a (quantum) reservoir.
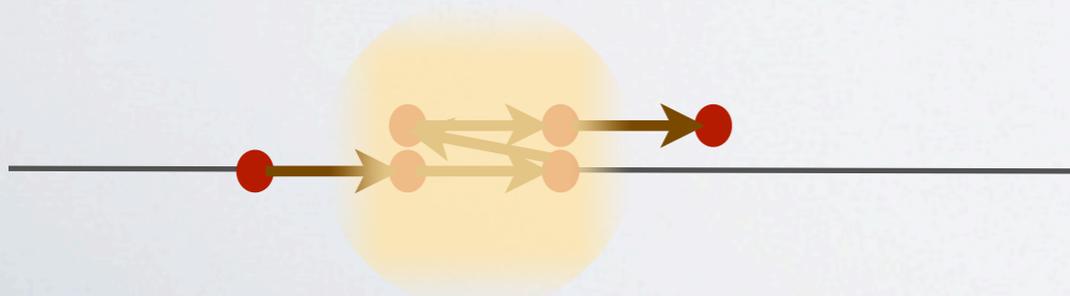
Usually:
Sum, or trace over, or average, over the reservoir degree of freedom, and look for an effective description of the quantum particle dynamics.

Lost of (some) noise related information (and of the filtration)

*** Discrete (unitary) quantum random walks.**   Aharonov, Davidovich, Zagury, '93, and ......

A (quantum) particle on a line (say Z), coupled to a (quantum) coin.

-- The quantum coin can be in two different states, (+ or -).
-- The particle move to the left and to the right conditioned on the coin state.
-- Use the same quantum coin at each step.

Discrete «open» quantum random walks (I).

A (quantum) particle on a line (say Z), with internal degree of freedom,
coupled to different quantum coins at each time step (recursive interactions).
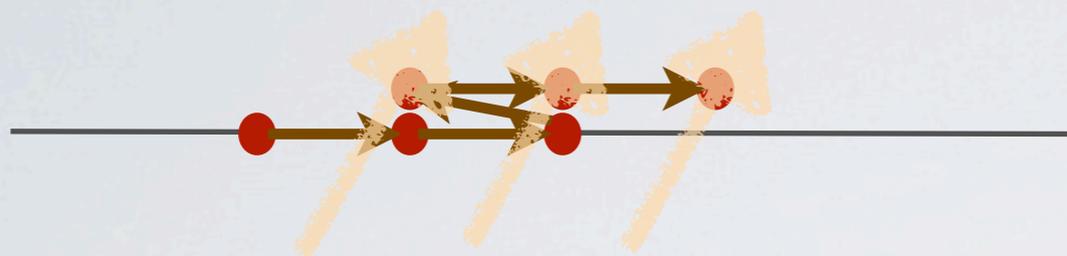
Attal, Petruccione, Sabot, Sinayskiy, 2012.

Actually a slight extension of their definition...

The quantum coin can be in two different states, (+ or -).
The particle move to the left and to the right conditioned on the coin state:
-- if the coin is measured in state +, the particle move one step to the right.
-- if the coin is measured in state -, it moves to the left.
But the coins act on the internal degrees of freedom (gyroscope),
And the probability to go left/right depends on this action.

* Hilbert space of states:
$\mathcal{H}_c \otimes \mathcal{H}_o \otimes \mathcal{H}_p^{\otimes\infty}$

$$\mathcal{H}_c = \mathbb{C}^2, \ \mathcal{H}_o = \mathbb{C}^{\mathbb{Z}}$$

$$\mathcal{H}_p^{\otimes\infty} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots$$

Coins are reset at each step.

p=probes (or coins);
c=color (or spin), internal d.o.f's;
o=position, orbital d.o.f.'s.

--- Notion of noise (with a measure, the probe state).
--- Notion of arrow of time (information on the probes)
--- Reservoir, by summing over the probes.

* Discrete «open» quantum random walks (II).

* Interaction (without measurements):

On state $|\psi\rangle_c \otimes |n\rangle_o \otimes |\phi\rangle_p$ in $\mathcal{H}_c \otimes \mathcal{H}_o \otimes \mathcal{H}_p$, the unitary evolution

$$(B_+ |\psi_c\rangle \otimes |n+1\rangle_o \otimes |+\rangle_p + (B_- |\psi\rangle_c) \otimes |n-1\rangle_o \otimes |-\rangle_p$$

Iterating interactions with different probes produces an «entangled» state, sum of states each indexed by a random walk (quantum parallelism).

* Measurements and «quantum trajectories»:

Measuring the probes, one may find + or - with probabilities, and then «project» the state on |+> or |->:

$$\propto (B_\pm |\psi\rangle_c) \otimes |n \pm 1\rangle_o \qquad \text{with probability} \qquad {}_c\langle\psi| B_\pm^\dagger B_\pm |\psi\rangle_c$$

Iterating gives random states indexed by random walks. The events, the output of the probe measurements, are in one-to-one correspondence with random walks.

* The original definition of open QRW was the mean of this process (not keeping track of the measurements).

* *«Open» quantum random walks trajectories:*

These are classical random processes with values on the line X the internal states.

(I) For pure (internal) state:

> If after n step, $(|\psi_n\rangle_c,\ x_n)$
>
> The updating is: $(|\psi_{n+1}\rangle_c = p_\pm(n)^{-1/2}\, B_\pm|\psi_n\rangle_c,\quad x_{n+1} = x_n \pm 1)$
>
> with probability: $p_\pm(n) = {}_c\langle\psi_n|B_\pm^\dagger B_\pm|\psi_n\rangle_c$

(I) For mixed (internal) state: ie. the system is not described by a vector but by a density matrix (a positive hermitian normalised matrix).

> If after n step, $(\rho_n,\ x_n)$
>
> The updating is: $(p_\pm(n)^{-1}\, B_\pm\rho_n B_\pm^\dagger,\quad x_n \pm 1)$
>
> with probability: $p_\pm(n) = \mathrm{tr}(B_\pm\rho_n B_\pm^\dagger)$

--- The matrices B are the moduli parameters of the walks.
--- The random process is now classical (but vector valued).
--- The internal system acts as a random (quantum) gyroscope.
    The position is slave to the probe measurements.
--- Events are random walks (output measurements),
    but with probabilities induced by the gyroscope motion.

Open Quantum Brownian motion:

--- Quantum trajectories: | Probes are measured (with random outputs), and the system density matrix evolves randomly (according to the measurement outputs).

more below.....

.... and this can be generalised with many packets and/or entangled states.

--- Quantum dynamical map: | Probes (reservoir) are *not* measured but trace out, and the system density matrix evolves with Lindblad equation

$$\partial_t \bar{\rho}_t = -\tfrac{1}{2}[P, [P, \bar{\rho}_t]] + i\big(N[P, \bar{\rho}_t] + [P, \bar{\rho}_t]N^\dagger\big) + i[H, \bar{\rho}_t] + L_N(\bar{\rho}_t)$$

for the density matrix $\quad \bar{\rho}_t := \int dx\, \rho(x, t) \otimes |x\rangle_o \langle x|$

--- Quantum Stochastic equation: | Probes (reservoir) are *not* measured but *not* trace out, and the *total* state evolves with a quantum-SDE.

$$dA_t = i[P + iN^\dagger, A_t]\, d\xi_t + i[P - iN, A_t]\, d\xi_t^\dagger + L_*(A_t)\, dt$$

for observable A, and dual Linbladian L.,
and (quantum) noises: $\quad d\xi_t\, d\xi_t^\dagger = dt,\ \ d\xi_t^\dagger\, d\xi_t = 0$
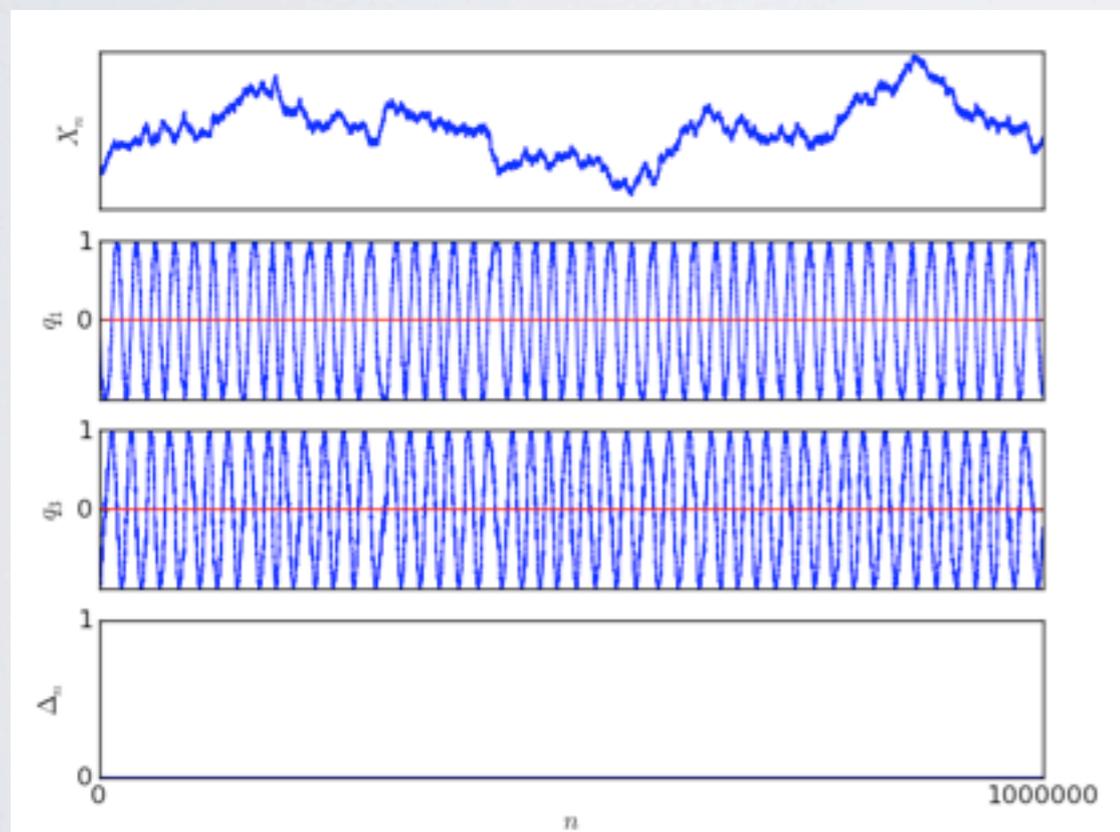
Discrete numerical simulations:

Unitarity or normalisation of probabilities imposes: $\qquad B_+^\dagger B_+ + B_-^\dagger B_- = \mathbb{I}$

A choice for the simulations: $\qquad B_+ = \delta^{-1}\begin{pmatrix} u & r \\ s & v \end{pmatrix}$ and $\qquad B_- = \delta^{-1}\begin{pmatrix} -v & s \\ r & -u \end{pmatrix}$
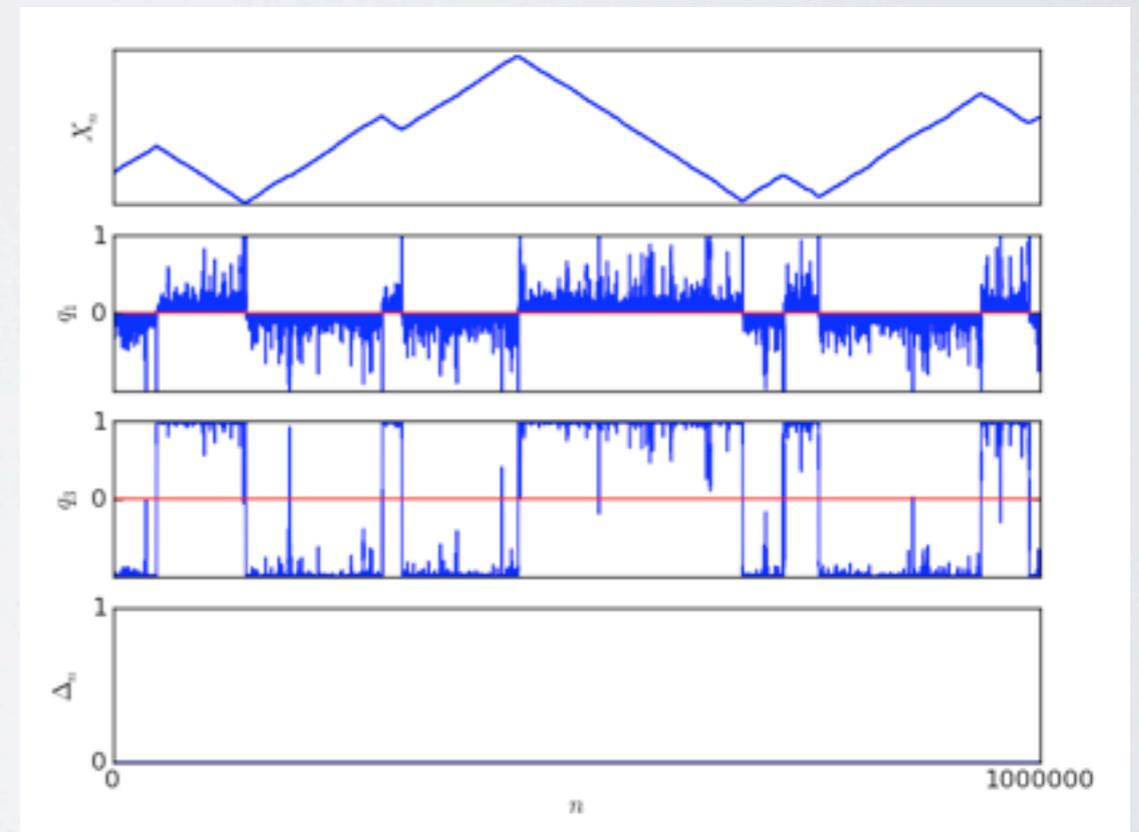
with $\qquad \delta = \sqrt{u^2 + v^2 + r^2 + s^2}$

A Brownian like regime.

$u = 1.005,\ v = 1.00$ and $r = -s = 0.00015$

A ballistic like (but diffusive) regime.

$u = 1.1,\ v = 1.00$ and $r = -s = 0.00015$

Kramer's like transition for a particle in a double well potential (… but not quite).

Projection a.s. on pure states:

As seen from the numerics, the determinant is vanishes (quickly).

--- Claim:
«Whatever initial value, the internal density matrix converges to pure states.»

Proof: $\rho_{n+1} = p_{\pm}(n)^{-1} B_{\pm} \rho_n B_{\pm}^{\dagger}$ with probability $p_{\pm}(n) = \mathrm{tr}(B_{\pm} \rho_n B_{\pm}^{\dagger})$

Let $\Delta_n := \det \rho_n$ be the determinant of the internal density matrix.

Then: $\mathbb{E}[\Delta_n^{1/2}] = c^n \, \Delta_0^{1/2}$ with $c := \det^{1/2}(M_+^2) + \det^{1/2}(M_-^2) < 1$,
and $\lim_{n \to \infty} \mathbb{E}[\Delta_n^{1/2}] = 0$.

Actually, Δ_n is a bounded sub-martingale and as such $\lim_{n \to \infty} \Delta_n = 0$
it converges almost surely and in L1.

--- This is true for the internal system alone (with or without orbital d.o.f.),
i.e. also true for a Q-bit coupled to series of Q-bits.

--- Similar to progressive collapse in non-demolition measurement
(Haroche's group experiments) but the target states keep involving randomly.

Scaling limit: «Open quantum Brownian motion».

As for classical Brownian motion, scale time, distance and moduli simultaneously: (New)

$$B_\pm = \tfrac{1}{\sqrt{2}}[\mathbb{I} \pm \sqrt{\epsilon}N + \epsilon(iH_\pm \pm M - \tfrac{1}{2}N^\dagger N) + o(\epsilon)]$$

with ϵ a small parameter and $H_\pm$, M hermitian but not N.

In the scaling limit one gets a time continuous process (with continuous measurement)

The scaling limit is $\epsilon \to 0$, $t = n\epsilon$ fixed, and $dx^2 \sim dt$.

$$\begin{aligned}
d\rho_t &= \big(i[H, \rho_t] + L_N(\rho_t)\big)dt + D_N(\rho_t)\, dB_t, \\
dX_t &= U_N(\rho_t)\, dt + dB_t,
\end{aligned}$$

A Brownian motion, coding for all probe measurements.

with $L_N(\rho) := N\rho N^\dagger - \tfrac{1}{2}(N^\dagger N\rho + \rho N^\dagger N)$

$D_N(\rho) := N\rho + \rho N^\dagger - \rho\, U_N(\rho)$ and $U_N(\rho_t) := \mathrm{tr}(N\rho + \rho N^\dagger)$

Both the density matrix and the position have stochastic evolution
but the position drift is guided by the internal gyroscope.

Proof: decompose the process as the sum of martingale plus a predictable process, i.e. a Doob decomposition.

$$M_n = \sum_{k=1}^{n} \pi_k \text{ with } \pi_k := \rho_k - \mathbb{E}[\rho_k|\mathcal{F}_{k-1}] \text{ and } O_n := \rho_n - M_n$$

--- This is for one trajectory, but multiple trajectory are ok (non localised initial wave packets)
--- Averaging over the noise: «open quantum brownian motion»
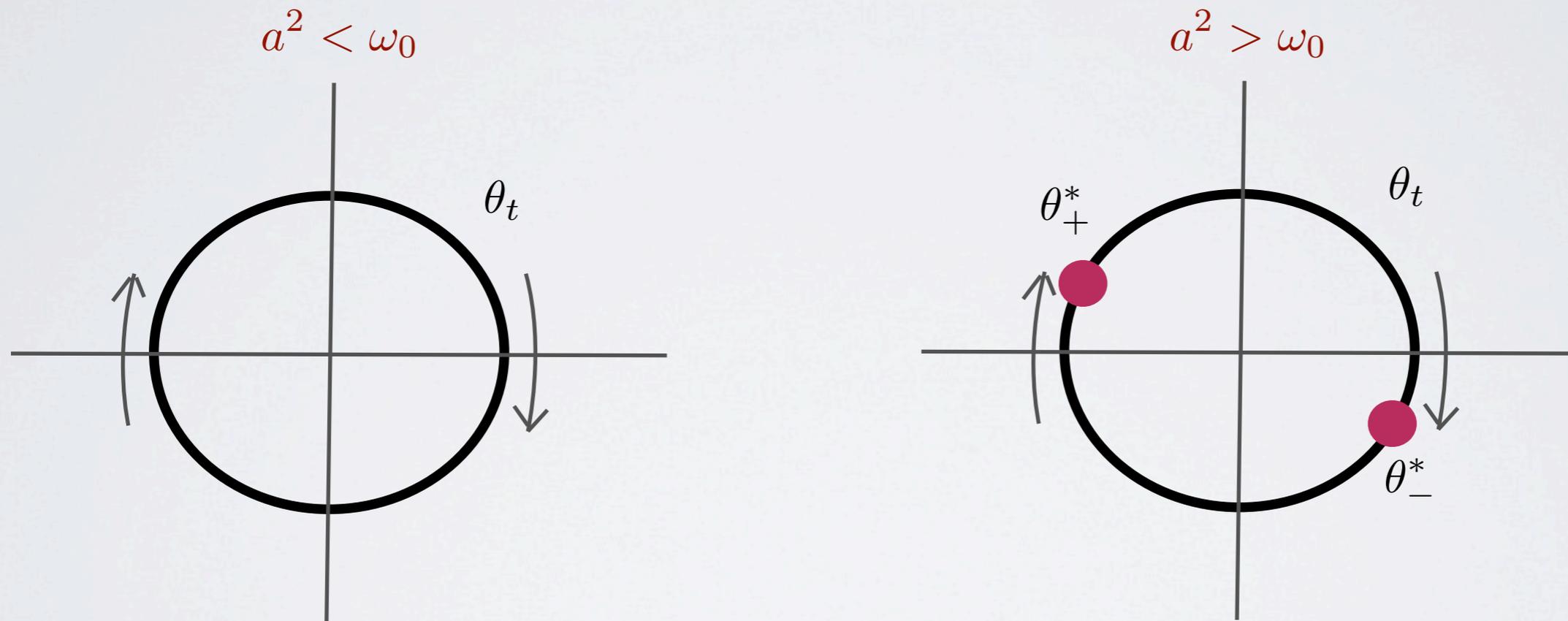   in contact with a reservoir.

Transition between two regimes:

Take $H := \omega_0 \, \sigma^2$ and $N = a \, \sigma^3$ with $\sigma^{1,2,3}$ the usual Pauli matrices.

Describe ρ_t as a pure state: $\rho_t = \frac{1}{2}(\mathbb{I} + q_1 \sigma^1 + q_3 \sigma^3)$ with $q_1 = \sin\theta$, $q_3 = \cos\theta$.

Eqs. $d\rho_t = \big(i[H, \rho_t] + L_N(\rho_t)\big)dt + D_N(\rho_t)\,dB_t$, becomes:

$$d\theta_t = -2(\omega_0 + a^2 \sin\theta_t \cos\theta_t)dt - 2a \sin\theta_t \, dB_t$$

$a^2 < \omega_0$

θ_t

$a^2 > \omega_0$

θ_+^*

θ_t

θ_-^*

Trajectories of theta are oriented : they never cross 0 and Pi anti-clockwise.

Two potential minima, with Kramer's transition between them. $a^2 > \omega_0$

Flips between two pure states:

The pure states are those associated to the two potential minima,
but need for a change of variable to see it.

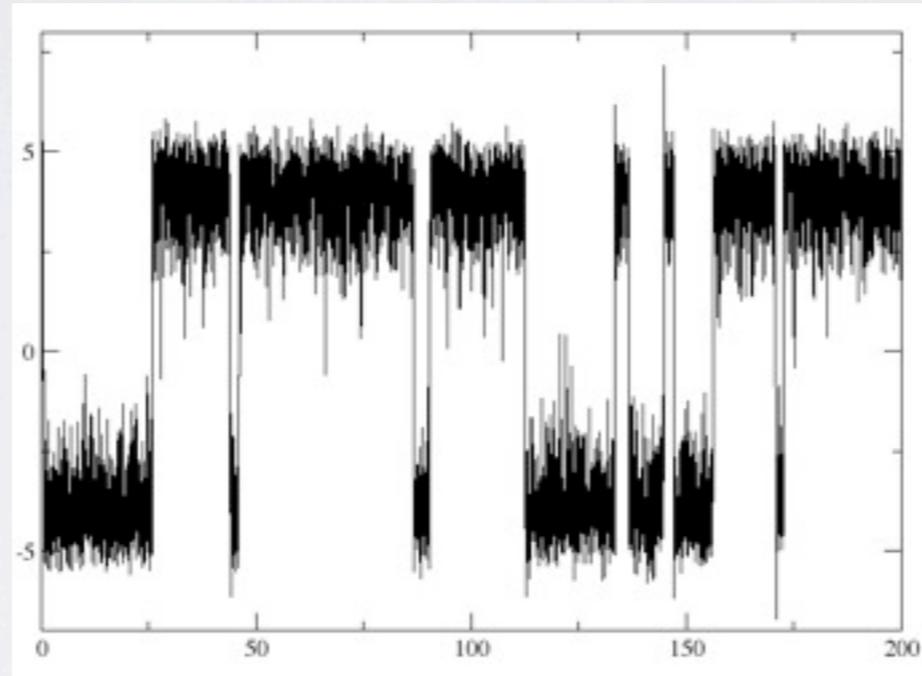Let $y_t := -\log|\tan\theta_t/2|$. It satisfies $dy_t = 2a\,dB_t - V'(y_t)dt$ with potential

$$V(y) = -2(\pm\omega_0 \sinh y + 2a^2 \log\cosh y)$$

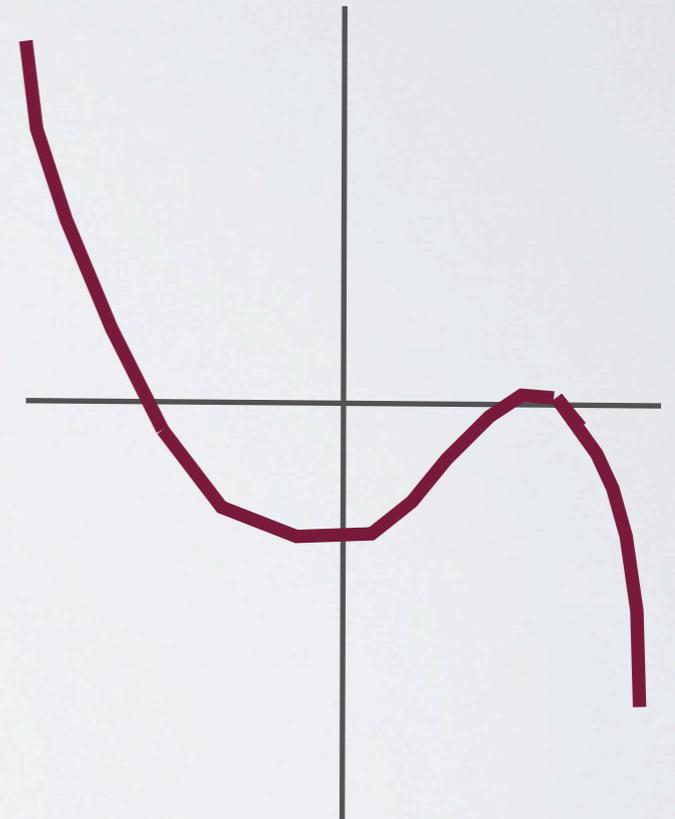A cubic like potential but with exponential ramps.

$a^2 < \omega_0$

$a^2 > \omega_0$

A y-trajectory
with Kramer's transition across the energy barrier.

No flips.

Flips:

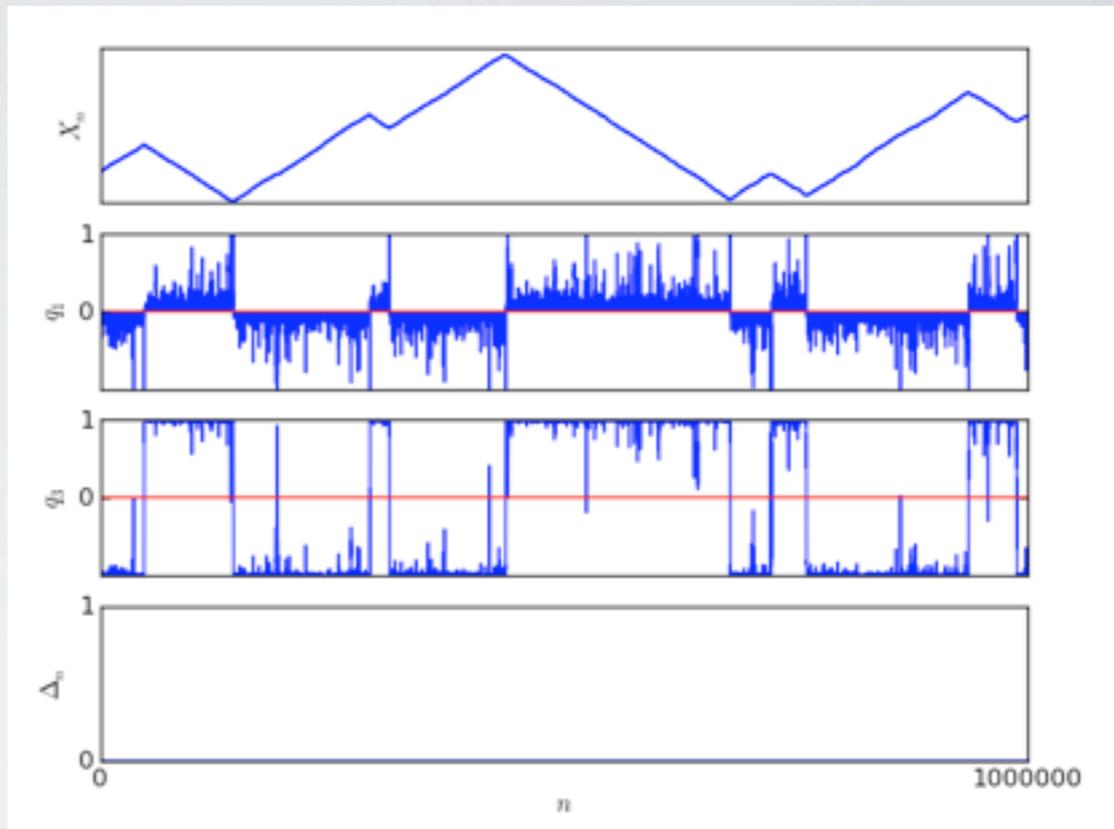$$\mathbb{E}[\tau_{\mathrm{flip}}] \simeq e^{\Delta V/4a^2} \simeq a^2$$

Ballistically induced diffusion:

$dX_t = U_N(\rho_t)\, dt + \, dB_t$ becomes:

$$dX_t = 2a \cos \theta_t \, dt + dB_t$$

with $2a \cos \theta_\pm^* \simeq \mp 2a$

Trajectories are ballistic,
with seesaw profiles induced
by gyroscope flips,
and large mean free paths.

But at very large time the position is Gaussian,
with large effective diffusion constant.

$$\mathbb{E}[X_t^2] = D_{\text{eff}}\, t \quad , \text{with } D_{\text{eff}} = 1 + 4a^4/\omega_0^2$$

Question: | Can we find similar phenomena (phenomenological description) producing a large effective diffusion constant (at large time)?

More on the continuous limit:

$$
\begin{aligned}
d\rho_t &= \big(i[H, \rho_t] + L_N(\rho_t)\big)dt + D_N(\rho_t)\, dB_t, \\
dX_t &= U_N(\rho_t)\, dt + dB_t,
\end{aligned}
$$

Proof: decompose the process as the sum of martingale plus a predictable process, i.e. a Doob decomposition.

$$
M_n = \sum_{k=1}^{n} \pi_k \text{ with } \pi_k := \rho_k - \mathbb{E}[\rho_k | \mathcal{F}_{k-1}] \text{ and } O_n := \rho_n - M_n
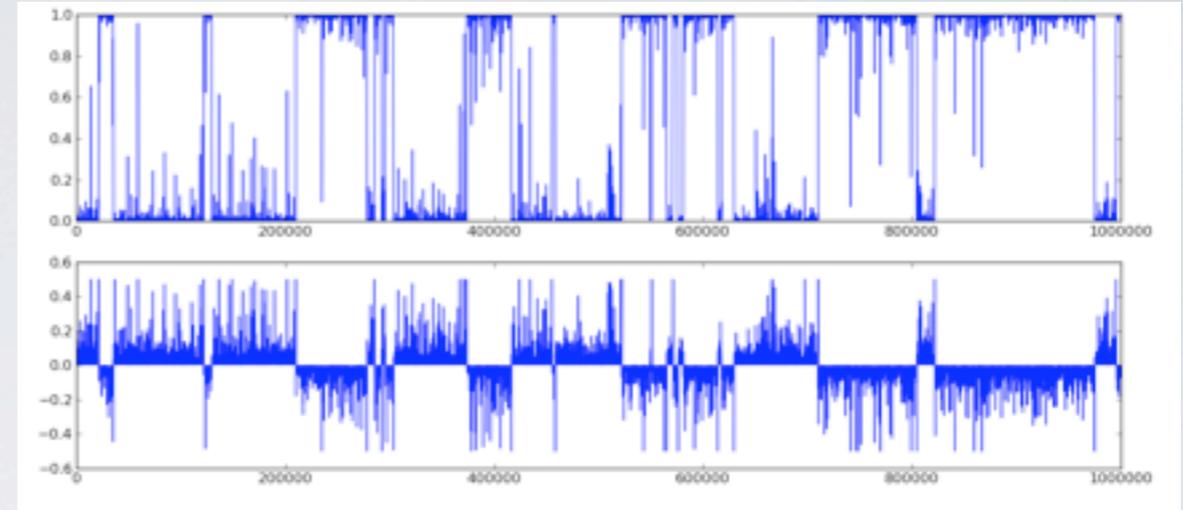$$

$$
\rho_{n+1} = \rho_n^{(+)} \mathbb{I}_{\{s_{n+1}=+\}} + \rho_n^{(-)} \mathbb{I}_{\{s_{n+1}=-\}} \quad \text{with} \quad \rho_n^{(\pm)} := B_\pm \rho_n B_\pm^\dagger / p_n^\pm
$$

$$
x_{n+1} - x_n = \mathbb{I}_{\{s_{n+1}=+\}} - \mathbb{I}_{\{s_{n+1}=-\}} \qquad p_n^\pm := \mathbb{E}[\mathbb{I}_{\{s_{n+1}=\pm\}} | \mathcal{F}_n] = \operatorname{tr}(B_\pm \rho_n B_\pm^\dagger)
$$

gives:
$$
2\pi_k = \big(\rho_k^{(+)} - \rho_k^{(-)}\big)\big(\mathbb{I}_{\{s_{k+1}=+\}} - p_k^+ + p_k^- - \mathbb{I}_{\{s_{k+1}=-\}}\big)
$$

Taylor expanding $dM_t := M_{n+1} - M_n$, $d\rho_t := \rho_{n+1} - \rho_n$ and $dX_t := \sqrt{\epsilon}(x_{n+1} - x_n)$. Identifying ϵ with dt, we get $dM_t = D_N(\rho_t)\, dB_t$,

More on bi-stability:

A system (spin half) under
continuous measurement.

-- The system (alias the gyroscope) is a Q-bit with hamiltonian $H = \omega_0\, \sigma^2$

with a (real) pure state: $\rho = \frac{1}{2}\left(1 + \cos\theta\sigma^3 + \sin\theta\sigma^1\right)$

-- In absence of measurement the system state oscillates.

-- Measuring an observable commuting with H : (progressive) collapse.

-- Measuring an observable not commuting with H : Q-jumps.

Generalizations:
--

More on the dynamical map:

$$\partial_t \bar{\rho}_t = -\tfrac{1}{2}[P,[P,\bar{\rho}_t]] + i\big(N[P,\bar{\rho}_t] + [P,\bar{\rho}_t]N^\dagger\big) + i[H,\bar{\rho}_t] + L_N(\bar{\rho}_t)$$

for the density matrix $\quad \bar{\rho}_t := \int dx\, \rho(x,t) \otimes |x\rangle_o\langle x|$

Proof:

From trajectory to mean density matrix, $\displaystyle\int dx\, \rho(x,t) \otimes |x\rangle_o\langle x| := \mathbb{E}[\,\rho_t \otimes |X_t\rangle_o\langle X_t|$

Then routine application of Ito calculus.

Computations are done by considering matrix elements $\int dx f^*(x)\rho(x,t)g(x)$ and identifying them with $f^*(X_t)g(X_t)\,\rho_t$ with X_t, ρ_t Q-trajectory.

$$\partial_t \rho(x,t) = \frac{1}{2}\partial_x^2\rho(x,t) - \big(N\partial_x\rho(x,t) + \partial_x\rho(x,t)N^\dagger\big) + i[H,\rho(x,t)] + L_N(\rho(x,t)),$$

This is equivalent to eq. above.
Generalisable in higher dimension and with in-homogenieties.

More on the SDE (open Brownian motion):

$$dA_t = i[P + iN^\dagger, A_t]\, d\xi_t + i[P - iN, A_t]\, d\xi_t^\dagger + L_*(A_t)\, dt$$

for observable A, and dual Linbladian L., and (quantum) noises: $\quad d\xi_t\, d\xi_t^\dagger = dt, \; d\xi_t^\dagger\, d\xi_t = 0$

A la physicist:

-- Noise $\quad d\xi_t := \int_t^{t+dt} a(s)ds, \quad [d\xi_t^\dagger, d\xi_t] = dt \quad$ with $\quad [a^\dagger(s), a(t)] = \delta(s - t)$

and a state, i.e. a measure (vacuum, Gibbs,...)

-- Look for quantum SDE: $\quad dA = D(A)d\xi_t + D^\dagger(A)d\xi_t^\dagger + L_*(A)dt$

with L* given (dual Lindbladian) and D(.) derivative.

with Ito rule $d\xi_t d\xi_t^\dagger = (1 + \gamma)\, dt$ and $d\xi_t^\dagger d\xi_t = \gamma\, dt$ for some γ.
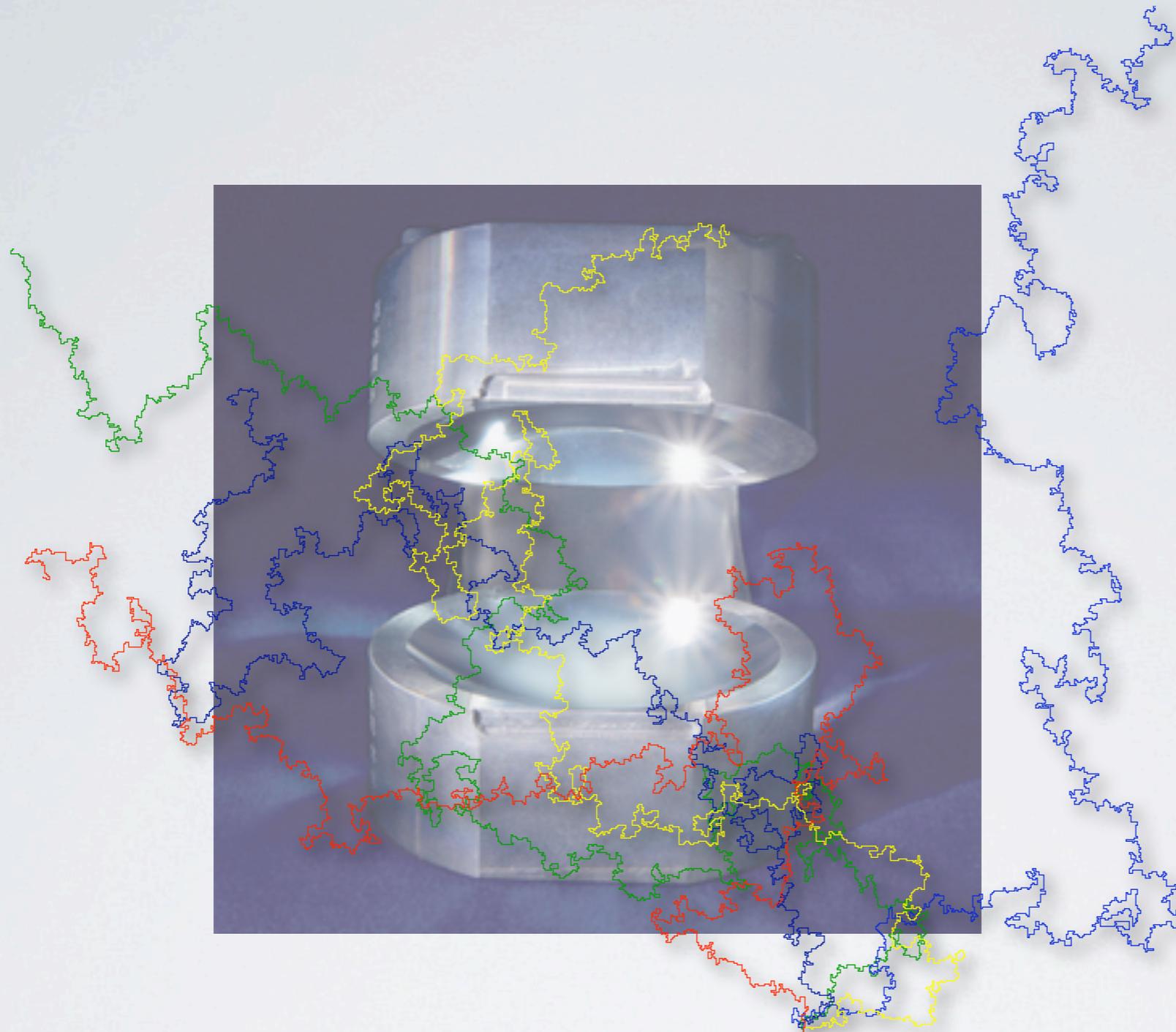
-- Consistency condition:

$$L_*(AB) = L_*(A)B + AL_*(B) - D(A)(1 + \gamma)D^\dagger(B) - D^\dagger(A)\gamma D(B)$$

I.e. L(.) second order differential operator and D(.) first order.
This determines D(.) and gamma.
(can be generalized with multi-noise).

Thank you.