Large time asymptotics of repeated interaction systems

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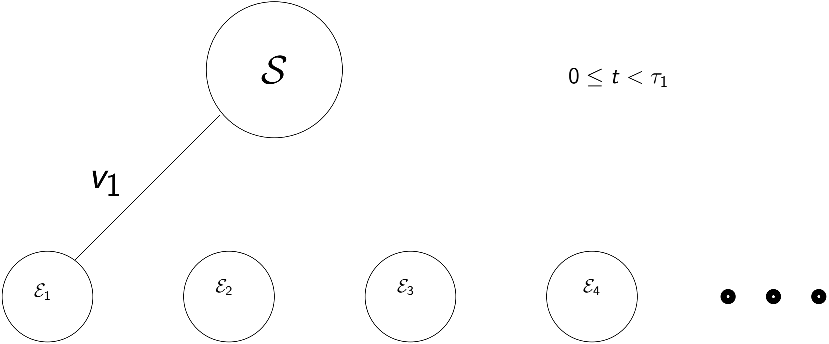
Repeated Interaction Systems (RIS)



$0 \leq t < \tau_1$

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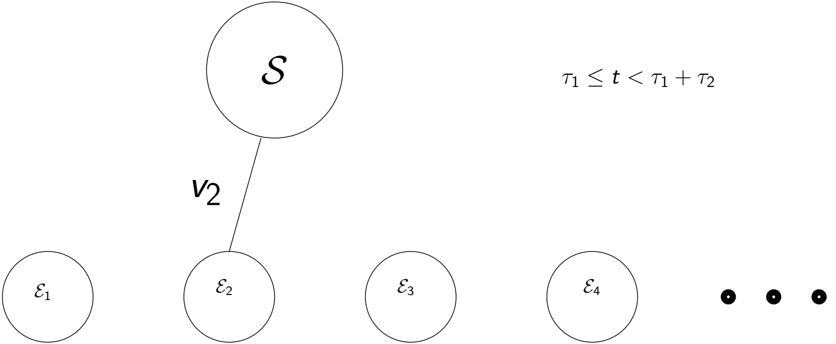

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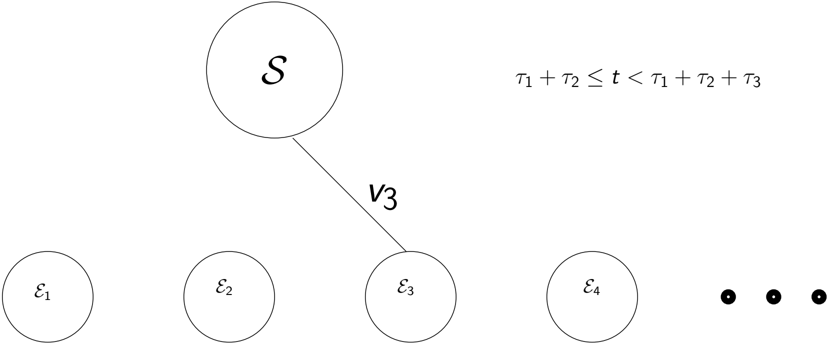

$$\tau_1 + \tau_2 \leq t < \tau_1 + \tau_2 + \tau_3$$

Hamiltonian description

A "small" system $\mathcal{S}$:

- Quantum system governed by some hamiltonian $h_{\mathcal{S}}$ acting on $\mathfrak{h}_{\mathcal{S}}$.

A chain $\mathcal{C}$ of quantum sub-systems $\mathcal{E}_k$ $(k = 1, 2, \ldots)$:

- $\mathcal{C} = \mathcal{E}_1 + \mathcal{E}_2 + \cdots$
- Each $\mathcal{E}_k$ is governed by some hamiltonian $h_{\mathcal{E}_k}$ acting on $\mathfrak{h}_{\mathcal{E}_k}$.

Interactions:

- Interaction operators v_k acting on $\mathfrak{h}_{\mathcal{S}} \otimes \mathfrak{h}_{\mathcal{E}_k}$.
- A sequence of interaction times $\tau_k > 0$.

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For $t \in [t_{n-1}, t_n[$, $t_n = \tau_1 + \cdots + \tau_n$:

- $\mathcal{S}$ interacts with $\mathcal{E}_n$,
- $\mathcal{E}_k$ evolves freely for $k \neq n$,

i.e. the full system is governed by

$$\tilde{h}_n = h_{\mathcal{S}} + h_{\mathcal{E}_n} + v_n + \sum_{k \neq n} h_{\mathcal{E}_k} = h_n + \sum_{k \neq n} h_{\mathcal{E}_k}.$$

The repeated interaction dynamics

Data:

1. Full Hamiltonian: $h_n = h_{\mathcal{S}} \otimes \mathbb{1}_{\mathcal{E}_n} + \mathbb{1}_{\mathcal{S}} \otimes h_{\mathcal{E}_n} + v_n$.
2. Initial state of $\mathcal{S}$: density matrix $\rho \in \mathcal{J}_1(\mathfrak{h}_{\mathcal{S}})$.
3. Initial state of $\mathcal{E}_n$: $\rho_{\mathcal{E}_n}$ (invariant state for the free dynamics of $\mathcal{E}_n$, e.g. Gibbs state at some inverse temperature β_n).

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After 0 interaction, the state of the total system is

$$\rho^{\mathrm{tot}}(0) := \qquad \rho \otimes \bigotimes_{k\geq 1} \rho_{\mathcal{E}_k}$$

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After 1 interaction, the state of the total system is

$$\rho^{\mathrm{tot}}(1) := \qquad \mathrm{e}^{-i\tau_1 \tilde{h}_1} \left(\rho \otimes \bigotimes_{k \geq 1} \rho_{\mathcal{E}_k} \right) \mathrm{e}^{i\tau_1 \tilde{h}_1}$$

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After 2 interactions, the state of the total system is

$$\rho^{\mathrm{tot}}(2) := \mathrm{e}^{-i\tau_2\tilde{h}_2}\mathrm{e}^{-i\tau_1\tilde{h}_1}\left(\rho \otimes \bigotimes_{k\geq 1} \rho_{\mathcal{E}_k}\right)\mathrm{e}^{i\tau_1\tilde{h}_1}\mathrm{e}^{i\tau_2\tilde{h}_2}$$

The repeated interaction dynamics

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1. Full Hamiltonian: $h_n = h_\mathcal{S} \otimes \mathbb{1}_{\mathcal{E}_n} + \mathbb{1}_\mathcal{S} \otimes h_{\mathcal{E}_n} + v_n$.
2. Initial state of $\mathcal{S}$: density matrix $\rho \in \mathcal{J}_1(\mathfrak{h}_\mathcal{S})$.
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After n interactions, the state of the total system is

$$\rho^{\mathrm{tot}}(n) := \mathrm{e}^{-i\tau_n \tilde{h}_n} \cdots \mathrm{e}^{-i\tau_2 \tilde{h}_2} \mathrm{e}^{-i\tau_1 \tilde{h}_1} \left(\rho \otimes \bigotimes_{k\geq 1} \rho_{\mathcal{E}_k} \right) \mathrm{e}^{i\tau_1 \tilde{h}_1} \mathrm{e}^{i\tau_2 \tilde{h}_2} \cdots \mathrm{e}^{i\tau_n \tilde{h}_n}.$$

The reduced dynamics map

We are interested in the system $\mathcal{S}$, i.e. (mainly) expectation values of observables of the form $A_{\mathcal{S}} \otimes \mathbb{1}$. At "time" n the state of $\mathcal{S}$ is given by

$$\rho(n) = \mathrm{Tr}_{\mathcal{C}}(\rho^{\mathrm{tot}}(n)).$$

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If $\mathcal{S}$ is in the state ρ before the n-th interaction, after it it is in the state

$$\mathcal{L}_n(\rho) := \mathrm{Tr}_{\mathcal{E}_n}\left(\mathrm{e}^{-i\tau_n h_n} \rho \otimes \rho_{\mathcal{E}_n} \mathrm{e}^{i\tau_n h_n}\right).$$

and the "repeated interaction" structure induces a markovian behavior:

$$\forall n, \quad \rho(n) = \mathcal{L}_n(\rho(n-1)).$$

$\Longrightarrow$ One has to understand $\mathcal{L}_n \circ \cdots \circ \mathcal{L}_1$ as $n \to \infty$.

Some questions about RIS

Large time behaviour:

- Existence of the limit $\lim_{n \to +\infty} \rho(n) = \rho_+$?
- Several situations : ideal (identical interactions, equilibrium), random (non-equilibrium), leaky (non-equilibrium).

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- Energy variation (external work, power delivered to the system)?
- In the non equilibrium case : fluxes?
- Entropy production?

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Concrete examples

Spectrum of a RDM

A key ingredient will be the spectral analysis of the RDM : existence of invariant state, spectral gap,... For example, in the ideal case $\rho(n) = \mathcal{L}_n \circ \cdots \circ \mathcal{L}_1(\rho) = \mathcal{L}^n(\rho)$.

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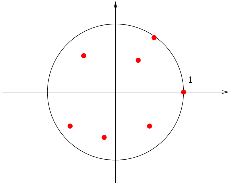
The $\mathcal{L}_n$ are completely positive and trace preserving maps on $\mathcal{J}_1(\mathfrak{h}_{\mathcal{S}})$.

Consequence:
$\mathrm{Spec}(\mathcal{L}_n) \subset \{z \in \mathbb{C} \,|\, |z| \leq 1\}$,
1 is in the spectrum.

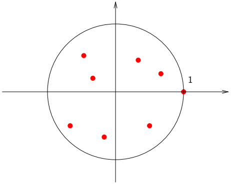


Ideal RIS

Take all the interactions identical, i.e. $\mathfrak{h}_{\mathcal{E}_k} \equiv \mathfrak{h}_{\mathcal{E}}$, $h_{\mathcal{E}_k} \equiv h_{\mathcal{E}}$, $\tau_k \equiv \tau$, $v_k \equiv v$. Hence $\mathcal{L}_k \equiv \mathcal{L}$.

Ergodic assumption (E):

$\mathrm{Spec}(\mathcal{L}) \cap S^1 = \{1\}$,
1 is a simple eigenvalue.



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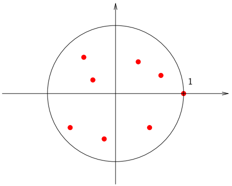

Theorem (B.-Joye-Merkli '06)

Let dim $\mathfrak{h}_{\mathcal{S}} < \infty$. If (E) is satisfied, there exist $C, \alpha > 0$ s.t. for any initial state ρ

$$\|\rho(n) - \rho_+\|_1 \leq C\mathrm{e}^{-\alpha n}, \qquad \forall n \in \mathbb{N},$$

where ρ_+ is the (unique) invariant state of $\mathcal{L}$.

Remark: one can allow for more general $\mathcal{S}$, assuming e.g. that 1 is an eigenvalue of $\mathcal{L}$ + spectral gap.

Ideal RIS: examples

Example 1:

- $\mathcal{S}$ and $\mathcal{E}_n$ are 2-level systems, i.e. $\mathfrak{h}_{\mathcal{S}} = \mathfrak{h}_{\mathcal{E}_n} \equiv \mathfrak{h}_{\mathcal{E}} = \mathbb{C}^2$, with energy levels $\{0, E\}$, resp. $\{0, E_0\}$, i.e. $h_{\mathcal{S}} = E a^* a$, $h_{\mathcal{E}_n} = E_0 b_n^* b_n$.
- $v_n = \lambda(a \otimes b_n^* + a^* \otimes b_n)$.
- $\rho_{\mathcal{E}_n} = \rho_{\mathcal{E},\beta} := \mathrm{e}^{-\beta h_{\mathcal{E}}} / \mathrm{Tr}(\mathrm{e}^{-\beta h_{\mathcal{E}}})$.

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If $\nu\tau \notin 2\pi\mathbb{N}$ where $\nu = \sqrt{(E-E_0)^2 + 4\lambda^2}$,

$$\|\rho(n) - \rho_{\mathcal{S},\beta^*}\|_1 \le C\mathrm{e}^{-\gamma n}, \qquad \forall n \in \mathbb{N},$$

where $\beta^* = \frac{E_0}{E}\beta$ and $\gamma = -\log\left(\sqrt{1 - \frac{4\lambda^2}{\nu^2}\sin^2\left(\frac{\nu\tau}{2}\right)}\right)$.

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Example 2: One-atom maser [B.-Pillet '09].
Same as above but $\mathfrak{h}_\mathcal{S} = \Gamma_+(\mathbb{C})$ and $h_\mathcal{S} = Ea^*a$ (single-mode EM field in a cavity). If, for any $k \in \mathbb{N}^*$, $\sqrt{(E-E_0)^2 + 4k\lambda^2}\tau \notin 2\pi\mathbb{N}$ then

$$\lim_{N\to\infty} \left\| \frac{1}{N}\sum_{n=1}^{N} \rho(n) - \rho_{\mathcal{S},\beta^*} \right\|_1 = 0$$

Random RIS: setup

We allow some fluctuations w.r.t. ideal situation (e.g. interaction time, temperature): $\mathcal{L} = \mathcal{L}(\omega_0)$ random variable with values in CP trace preserving maps on $\mathfrak{h}_{\mathcal{S}}$ over some probability space $(\Omega_0, \mathcal{F}, \mathrm{p})$.

Product of i.i.d. RDMs: $\Omega = \Omega_0^{\mathbb{N}^*}$, $\mathrm{d}\mathbb{P} = \prod_{n\geq 1} \mathrm{dp}$ and $\omega = (\omega_n)_{n\geq 1}$.
$\Rightarrow$ Understand $\rho(n,\omega) = (\mathcal{L}(\omega_n) \circ \cdots \circ \mathcal{L}(\omega_1))(\rho)$.

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A simple case: $\mathcal{L}(\omega_0)$ is a rank one projection, i.e. $(\mathcal{L}(\omega_0))(\rho) \equiv \rho_+(\omega_0)$.
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$\Rightarrow$ For any n, $\rho(n,\omega) = \rho_+(\omega_n)$.

Consequence: unless $\rho_+(\omega_0) \equiv \rho_+$, no convergence in the usual sense, but in the ergodic mean

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \rho(n,\omega) = \mathbb{E}(\rho_+), \quad a.e.\ \omega.$$

Random RIS: results

Theorem (B.-Joye-Merkli '08)

Let dim $\mathfrak{h}_\mathcal{S} < \infty$. *If* $\mathrm{p}(\mathcal{L}(\omega_0)$ *satisfies (E))* > 0, *then*

1. $\mathbb{E}(\mathcal{L})$ *satisfies (E).*
2. *For any* $\rho \in \mathcal{J}_1(\mathfrak{h}_\mathcal{S})$, $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \rho(n,\omega) = \rho_+$, *a.e.* $\omega \in \Omega$, *where* ρ_+ *is the unique invariant state of* $\mathbb{E}(\mathcal{L})$.

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If moreover there exists ρ_+ *s.t.* $\mathcal{L}(\omega_0)(\rho_+) = \rho_+$ *for a.e.* ω_0, *i.e. there is a deterministic invariant state, then there exists* $\alpha > 0$ *s.t. for any* $\rho \in \mathcal{J}_1(\mathfrak{h}_{\mathcal{S}})$ *and for a.e.* $\omega \in \Omega$, *there exists* $C(\omega) > 0$

$$\|\rho(n,\omega) - \rho_+\|_1 \leq C(\omega)\mathrm{e}^{-\alpha n}, \quad \forall n \in \mathbb{N}.$$

Remark: it is a law of large numbers.
Open question: more information on the law of $\rho(n,\omega)$?

Random RIS: example

Take example 1 above (2-level systems). Recall:

- $\mathcal{L}$ satisfies (E) iff $\tau \notin T\mathbb{N}$ with $T = 2\pi/\sqrt{(E-E_0)^2+4\lambda^2}$,
- $\rho_{\mathcal{S},\beta^*}$ is an invariant state of $\mathcal{L}$, with $\beta^* = \beta E_0/E$.

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2. Suppose $\tau_n \equiv \tau \notin T\mathbb{N}$ and $\beta(\omega)$ is a random variable. Then for any $\rho \in \mathcal{J}_1(\mathcal{H}_\mathcal{S})$,

$$\lim_{N\to\infty} \sum_{n=1}^{N} \rho(n,\omega) = \mathbb{E}(\rho_{\mathcal{S},\beta^*(\omega)}).$$

Leaky RIS: setup

The system $\mathcal{S}$ is now coupled to an additional reservoir $\mathcal{R}$ (e.g. to take into account the losses in the cavity of the one-atom maser).

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The additional reservoir consists in a gas of non-interacting and non-relativistic fermions at inverse temperature $\beta_{\mathcal{R}}$:

- one particle space: $\mathfrak{h}_{\mathcal{R},1} = L^2(\mathbb{R}^3, \mathrm{d}k)$.
- one particle hamiltonian: $h_{\mathcal{R},1} = |k|^2$.
- The total Hilbert space is then $\mathfrak{h}_{\mathcal{R}} = \Gamma_-(\mathfrak{h}_{\mathcal{R},1}) = \oplus_{n=0}^{\infty} \wedge^n \mathfrak{h}_{\mathcal{R},1}$.
- The hamiltonian is $h_{\mathcal{R}} = \mathrm{d}\Gamma(h_{\mathcal{R},1})$.
- Its initial state $\rho_{\mathcal{R}}$ is the quasi-free state with one-body density matrix $T_{\mathcal{R}} = (1 + \mathrm{e}^{\beta_{\mathcal{R}} h_{\mathcal{R},1}})^{-1}$, i.e. $\rho_{\mathcal{R}}(a^*(f)a(g)) = \langle g, T_{\mathcal{R}} f \rangle$.

Leaky RIS: result

$\Rightarrow$ "$h = h_{\mathcal{S}} + h_{\mathcal{E}} + h_{\mathcal{R}} + \lambda_{\mathcal{E}} v_{\mathcal{E}} + \lambda_{\mathcal{R}} v_{\mathcal{R}}$".

$\mathcal{L}$ is now the RDM with $\mathcal{S}$ replaced by $\mathcal{S} + \mathcal{R}$.

Issue: one cannot apply the previous results since the spectral assumption (E) is typically not satisfied (essential spectrum on the unit circle due to the presence of $\mathcal{R}$).

Solution: use a deformed version $\mathcal{L}_\theta := \mathcal{T}_\theta \circ \mathcal{L} \circ \mathcal{T}_\theta^{-1}$ of $\mathcal{L}$ (analytic deformation in the reservoir variables) to get rid of essential spectrum.

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Theorem (B.-Joye-Merkli '10)

Let dim $\mathfrak{h}_{\mathcal{S}} < \infty$ + regularity assumptions. If $\mathcal{L}_\theta$ satisfies (E) for some θ, there exists $\lambda_0 > 0$ s.t. if $0 < |\lambda_{\mathcal{R},\mathcal{E}}| < \lambda_0$ for any initial state ρ of $\mathcal{S}$ and any observable A on $\mathcal{S} + \mathcal{R}$

$$\lim_{n\to\infty} \mathrm{Tr}\left(\mathcal{L}^n(\rho \otimes \rho_{\beta_{\mathcal{R}}})A\right) = \rho_{+,\lambda,\theta}(A(\theta)) =: \rho_{+,\lambda}(A),$$

where $\rho_{+,\lambda,\theta}$ is the (unique) invariant state of $\mathcal{L}_\theta$ and $A(\theta) = \mathcal{T}_\theta(A)$ is the analytic deformation of A.

Leaky RIS: example

We continue example 1:

- $\mathcal{S}$ and the $\mathcal{E}_n$'s are 2-level systems with energy levels $\{0, E\}$, resp. $\{0, E_0\}$.
- the $\mathcal{E}_n$'s are initially in thermal equilibrium at inverse temperature $\beta_{\mathcal{E}}$.
- $v_{\mathcal{R}} = \sigma_x \otimes (a(f) + a^*(f))$ where f is a form factor satisfying some regularity assumption.

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If $\|f(\sqrt{E})\|_{\mathfrak{G}}^2 := \int_{S^2} |f(\sqrt{E}\,\sigma)|^2 \mathrm{d}\sigma \neq 0$ and $\tau(E_0 - E) \notin 2\pi\mathbb{Z}^*$, then the above theorem applies and the asymptotic state $\rho_{+,\lambda}$ is given by

$$\rho_{+,\lambda} = \left(\gamma \rho_{\mathcal{S},\beta_{\mathcal{R}}} + (1-\gamma)\rho_{\mathcal{S},\beta_{\mathcal{E}}^*}\right) \otimes \rho_{\mathcal{R}} + O(\lambda),$$

where $\beta_{\mathcal{E}}^* = \frac{E_0}{E}\beta_{\mathcal{E}}$ and γ is given by

$$\gamma = \frac{\lambda_{\mathcal{R}}^2 \gamma_{\mathrm{th}}}{\lambda_{\mathcal{R}}^2 \gamma_{\mathrm{th}} + \lambda_{\mathcal{E}}^2 \gamma_{\mathrm{ri}}}, \quad \gamma_{\mathrm{th}} = \frac{\pi}{2}\sqrt{E}\|f(\sqrt{E})\|_{\mathfrak{G}}^2, \quad \gamma_{\mathrm{ri}} = \frac{\tau}{2}\mathrm{sinc}^2\left(\frac{\tau(E_0 - E)}{2}\right).$$

Energy variation

The total Hamiltonian is time-dependent $\Rightarrow$ the total energy is usually not conserved.
During the n-th interaction the energy is constant, formally given by

$$\mathrm{Tr}\left(\rho^{\mathrm{tot}}(n-1)h_n\right) = \mathrm{Tr}\left(\rho^{\mathrm{tot}}(n)h_n\right).$$

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When one switches from interaction n to interaction $n+1$, there is an energy jump (external work):

$$\begin{aligned} \delta W(n) &:= \mathrm{Tr}\big(\rho^{\mathrm{tot}}(n) \times (h_{n+1} - h_n)\big) = \mathrm{Tr}\big(\rho^{\mathrm{tot}}(n) \times (v_{n+1} - v_n)\big) \\ &= \mathrm{Tr}_{\mathcal{S},\mathcal{E}_{n+1}}\left[\rho(n) \otimes \rho_{\mathcal{E}_{n+1}} \; v_{n+1}\right] \\ &\qquad -\mathrm{Tr}_{\mathcal{S},\mathcal{E}_n}\left[\rho(n-1) \otimes \rho_{\mathcal{E}_n} \; \mathrm{e}^{\mathrm{i}\tau_n h_n} v_n \mathrm{e}^{-\mathrm{i}\tau_n h_n}\right]. \end{aligned}$$

Energy variation

In the ideal case, one easily gets

Proposition (B.-Joye-Merkli '06)

If Assumption (E) is satisfied,

$$\Delta W := \lim_{n\to\infty} \frac{1}{\tau}\delta W(n) = \frac{1}{\tau}\mathrm{Tr}_{\mathcal{S},\mathcal{E}}\left(\rho_+ \otimes \rho_{\mathcal{E}}\left(v - \mathrm{e}^{i\tau h} v \mathrm{e}^{-i\tau h}\right)\right).$$

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In the random case we have,

Proposition (B.-Joye-Merkli '08)

If $\mathrm{p}(\mathcal{L}(\omega_0)$ *satisfies (E))* > 0, *then*

$$\Delta W := \lim_{N\to\infty} \frac{1}{t_N(\omega)} \sum_{n=1}^{N} \delta W(n) = \frac{\mathbb{E}\left(\mathrm{Tr}_{\mathcal{S},\mathcal{E}}\left(\rho_+ \otimes \rho_{\mathcal{E}}\left(v - \mathrm{e}^{i\tau h} v \mathrm{e}^{-i\tau h}\right)\right)\right)}{\mathbb{E}(\tau)},$$

where ρ_+ *is the unique invariant state of* $\mathbb{E}(\mathcal{L})$.

Entropy production

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Relative entropy: $\mathrm{Ent}(\rho|\rho_0) = \mathrm{Tr}(\rho \log \rho - \rho \log \rho_0) \geq 0$.

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Theorem (B.-Joye-Merkli '06 -'08)

1) Ideal case: if (E) is satisfied, then

$$\Delta S := \lim_{n\to\infty} \frac{1}{\tau}(\mathrm{Ent}(\rho_{n+1}^{\mathrm{tot}}|\rho_0) - \mathrm{Ent}(\rho_n^{\mathrm{tot}}|\rho_0)) = \beta \Delta W.$$

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1) Ideal case: if (E) is satisfied, then

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2) Random case: if $\mathrm{p}(\mathcal{L}(\omega_0)$ satisfies (E)$) > 0$, then

$$\begin{aligned} \Delta S &:= \lim_{n\to\infty} \frac{\mathrm{Ent}(\rho^{\mathrm{tot}}_n|\rho_0) - \mathrm{Ent}(\rho|\rho_0)}{t_n(\omega)} \\ &= \frac{\mathbb{E}\left(\beta\, \mathrm{Tr}_{\mathcal{S},\mathcal{E}}\left(\rho_+ \otimes \rho_{\mathcal{E}}\left(v - \mathrm{e}^{i\tau h} v \mathrm{e}^{-i\tau h}\right)\right)\right)}{\mathbb{E}(\tau)}. \end{aligned}$$

In particular, if β is not random we still have $\Delta S = \beta \Delta W$.

Thermodynamics of leaky RIS

$\mathcal{S}$ is coupled to two environments : the chain and the reservoir. Besides ΔW and ΔS we can study the energy fluxes.

During the n-th interaction, the energy changes in the chain and the reservoir are

$$\begin{aligned}\delta E_{\mathcal{C}}(n) &:= \mathrm{Tr}\big((\rho^{\mathrm{tot}}(n+1) - \rho^{\mathrm{tot}}(n)) \times h_n\big),\\ \delta E_{\mathcal{R}}(n) &:= \mathrm{Tr}\big((\rho^{\mathrm{tot}}(n+1) - \rho^{\mathrm{tot}}(n)) \times h_{\mathcal{R}}\big).\end{aligned}$$

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Proposition (B.-Joye-Merkli '10)

Assume $\mathcal{L}_\theta$ satisfies (E) for some θ. Then the limits

$$\Delta E_{\mathcal{C}} := \lim_{N\to\infty} \frac{1}{N\tau} \sum_{n=1}^{N} \delta E_{\mathcal{C}}(n) \text{ and } \Delta E_{\mathcal{R}} := \lim_{N\to\infty} \frac{1}{N\tau} \sum_{n=1}^{N} \delta E_{\mathcal{R}}(n)$$

exist as well as ΔW and ΔS. Moreover we have

$$\Delta W = \Delta E_{\mathcal{C}} + \Delta E_{\mathcal{R}} \text{ and } \Delta S = \beta_{\mathcal{E}} \Delta E_{\mathcal{C}} + \beta_{\mathcal{R}} \Delta E_{\mathcal{R}}.$$

Thermodynamics of the example

Recall $T = 2\pi/\sqrt{(E - E_0)^2 + 4\lambda^2}$, and let $\kappa := \frac{16\pi^2\lambda^2 E_0}{T^2}\sin^2\left(\frac{\pi\tau}{T}\right)$.

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Without leak we compute explicitly

$$\Delta W = \frac{\mathrm{Cov}\left(\kappa, \frac{1}{1+\mathrm{e}^{-\beta E_{\mathcal{E}}}}\right)}{\mathbb{E}\left(\frac{1}{1+\mathrm{e}^{-\beta E_{\mathcal{E}}}}\right)}, \quad \Delta S = \frac{\mathrm{Cov}\left(\beta\kappa, \frac{1}{1+\mathrm{e}^{-\beta E_{\mathcal{E}}}}\right)}{\mathbb{E}\left(\frac{1}{1+\mathrm{e}^{-\beta E_{\mathcal{E}}}}\right)}.$$

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In particular, if only τ is random (or ideal situation) $\Delta W = \Delta S = 0$, and if only β is random $\Delta W = 0$ while $\Delta S \geq 0$ and vanishes iff $\beta(\omega) \equiv \beta$ a.s.
With leaks we have

$$\begin{aligned}
\Delta E_{\mathcal{C}} &= CE_0\left(\mathrm{e}^{-\beta_{\mathcal{R}}E} - \mathrm{e}^{-\beta^*_{\mathcal{E}}E}\right) + O(\lambda^3),\\
\Delta E_{\mathcal{R}} &= CE\left(\mathrm{e}^{-\beta^*_{\mathcal{E}}E} - \mathrm{e}^{-\beta_{\mathcal{R}}E}\right) + O(\lambda^3),\\
\Delta W &= C(E_0 - E)\left(\mathrm{e}^{-\beta_{\mathcal{R}}E} - \mathrm{e}^{-\beta^*_{\mathcal{E}}E}\right) + O(\lambda^3),\\
\Delta S &= C(\beta^*_{\mathcal{E}}E - \beta_{\mathcal{R}}E)\left(\mathrm{e}^{-\beta_{\mathcal{R}}E} - \mathrm{e}^{-\beta^*_{\mathcal{E}}E}\right) + O(\lambda^3),
\end{aligned}$$

where C is explicit and of order λ^2.