

*Spectral gap and (modified) log-Sobolev  
inequalities for quantum Markov semigroups*

Raffaella Carbone

Dipartimento di Matematica "F. Casorati", Università di Pavia

Autrans, july 8–19, 2013

# *Schedule*

## *Markov Semigroups*

Invariant states and associated norms

## *Spectral gap*

## *Log-Sobolev inequalities*

## *Mutual relations and entropy decay*

$\mathcal{A}$  is a von Neumann algebra  $\subseteq \mathcal{B}(\mathfrak{h})$  ( $\mathcal{A} = M_n(\mathbb{C})$ )

$\mathfrak{h}$  separable Hilbert space

$\mathcal{T}$  is a Quantum Markov Semigroup (QMS) on  $\mathcal{A}$ , i.e.  $\mathcal{T} = (\mathcal{T}_t)_{t \geq 0}$

is a family of linear operators  $\mathcal{T}_t : \mathcal{M} \rightarrow \mathcal{M}$  such that:

1.  $w^*$ - continuous semigroup ( $\mathcal{T}_0 = \mathbb{1}$ ,  $\mathcal{T}_{t+s} = \mathcal{T}_t \circ \mathcal{T}_s$ )
2.  $\mathcal{T}_t(\mathbb{1}) = \mathbb{1}$
3.  $\mathcal{T}_t$  completely positive

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$\Rightarrow$  each  $\mathcal{T}_t$  is a contractive operator



$\mathcal{A}$  is a von Neumann algebra  $\subseteq \mathcal{B}(h)$  ( $\mathcal{A} = M_n(\mathbb{C})$ )

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$\mathcal{L}$  infinitesimal generator,  $\mathcal{L} = \frac{d}{dt} \mathcal{T}_t|_{t=0}$

$\mathcal{E}$  sesquilinear form, ....



## *invariant states*

$\exists$  a faithful (normal) and invariant state  $\rho$  ( $\rho =$  its density matrix)

$\rho$  invariant  $\Leftrightarrow \text{tr}(\rho \mathcal{T}_t(f)) = \text{tr}(\rho f)$  for all  $f$  and  $t$

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induced scalar product and norms, ( $f, g \in \mathcal{A}, \rho \geq 1$ )

$$\langle f, g \rangle_\rho = \text{tr}(\rho^{1/2} f^* \rho^{1/2} g)$$

$$\|f\|_{p,\rho}^p = \text{tr}(|\rho^{1/2p} f \rho^{1/2p}|^p)$$



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$\mathcal{T}$  is  $\rho$ -symmetric if  $\langle f, \mathcal{T}_t g \rangle_\rho = \langle \mathcal{T}_t f, g \rangle_\rho$  for all  $f, g$  and  $t$   
 iff  $\langle f, \mathcal{L} g \rangle_\rho = \langle \mathcal{L} f, g \rangle_\rho$  for all  $f, y$

$\mathcal{E}(f, g) := -\langle f, \mathcal{L} g \rangle_\rho$        $\mathcal{E}$  is a sesquilinear positive form on  $\mathcal{A}$

(Analogue for commutative algebras)

## Spectral gap

$$\rho(f) = \text{tr}(\rho f) = \langle \mathbf{1}, f \rangle_\rho, \quad \text{Var}(f) = \|f - \rho(f)\|_{2,\rho}^2$$

$$\text{gap}(\mathcal{L}) := \inf \left\{ \frac{\mathcal{E}(f,f)}{\text{Var}(f)}, f \in \mathcal{A}, f \neq \rho(f) \right\} \geq 0$$

i.e.  $\text{gap}(\mathcal{L})$  is the best constant  $\lambda$  verifying

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**Th.**  $\text{gap}(\mathcal{L})$  is the best constant  $\lambda$  verifying

$$\|\mathcal{T}_t f - \rho(f)\|_2 \leq e^{-\lambda t} \|f - \rho(f)\|_2 \quad \text{for all } f \in \mathcal{A}, t \geq 0.$$

$$\mathcal{T}_t \mathbf{1} = \mathbf{1}, \quad Pr_{\mathbf{1}}(f) = \langle \mathbf{1}, f \rangle_\rho \mathbf{1} = \rho(f) \mathbf{1}$$

$$\|\mathcal{T}_t f - \rho(f)\|_2^2 = \|\mathcal{T}_t f - \rho(\mathcal{T}_t f)\|_2^2 = \text{Var}(\mathcal{T}_t f)$$

- the definition can be extended to the case when the invariant state is not unique replacing  $\text{span}\{\mathbb{1}\}$  with  $\ker \mathcal{L}$

$$\text{gap}(\mathcal{L}) := \inf \left\{ \frac{\mathcal{E}(f)}{\|f\|_2^2} : f \in (\ker \mathcal{L})^\perp \setminus \{0\} \right\}$$

- **Th.** (RC and J. Bolaños, 2013 – Reduction method)

$\mathcal{T}$  semigroup acting on a Hilbert space  $\mathcal{H}$  with a generator  $\mathcal{L}$ ,

$$\mathcal{H} = \bigoplus_{n \in N} \mathcal{H}_n, \quad N \text{ finite or denumerable}$$

where the subspaces  $(\mathcal{H}_n)_n$  are closed, mutually orthogonal and  $\mathcal{L}(\mathcal{H}_n) \subset \mathcal{H}_n$ . Then

- 1-  $f = \sum_n f_n$  with  $f_n \in \mathcal{H}_n \Rightarrow \mathcal{E}(f) = \sum_n \mathcal{E}(f_n)$ ,  $\|f\|_2^2 = \sum_n \|f_n\|_2^2$ ,
- 2- if, moreover,  $(\ker \mathcal{L}) = \bigoplus_n \mathcal{K}_n$ , with  $\mathcal{K}_n \subset \mathcal{H}_n$ , then

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## *Examples*

$$\mathcal{L}(x) = i[H, x] - \frac{1}{2} \sum_{j \geq 1} (L_j^* L_j x - 2L_j^* x L_j + x L_j^* L_j), \quad x \in \mathfrak{B}(\mathfrak{h})$$

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1.  $\mathfrak{h} = l^2(\mathbb{N})$ ,  $(e_n)_n$  o.n. canonical basis,

$$S e_n = e_{n+1}, \quad N e_n = n e_n, \quad a^+ = N^{1/2} S$$

1a- quantum birth and death semigroups:

$$L_1 = \mu(N)S, \quad L_2 = \lambda(N)S^*, \quad H = \dots$$

(a quantum Ornstein-Uhlenbeck  $L_1 = \mu_0 a$ ,  $L_2 = \lambda_0 a^+$  with  $\lambda_0 < \mu_0$ ,  
or 2-photon absorption and emission process

$$L_1 = \mu_0 a^2, L_2 = \lambda_0 a^{+2}, H = \kappa a^{+2} a^2)$$

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1b- generic semigroups

$$L_{mj} = \sqrt{\gamma_{mj}} |e_j\rangle \langle e_m| \quad \text{for } j \neq m, \quad H = \sum_{m \in \mathbb{N}} \kappa_m |e_m\rangle \langle e_m|,$$

with  $\gamma_{mj} \geq 0$  for  $m \neq j$ ,  $\gamma_{mm} := -\sum_{j \neq m} \gamma_{mj}$  finite,  $\kappa_m \in \mathbb{R}$

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2.  $\mathfrak{h} = \mathbb{C}^p$ , circulant semigroups

## *Log-Sobolev inequalities*

$f \in \mathcal{A}, f > 0$ , denote  $\tilde{f} = \rho^{1/4} f \rho^{1/4}$

$$H(f) = \operatorname{tr} \left( \tilde{f}^2 (\lg \tilde{f} - \frac{1}{2} \lg \rho) \right) - \|f\|_{2,\rho}^2 \lg \|f\|_{2,\rho}$$

if  $[\rho, f] = 0$ , then  $H(f) = \operatorname{tr}(\rho f^2 \lg \frac{f}{\|f\|_{2,\rho}}) = \sum_k \rho_k f_k^2 \lg \frac{f_k}{\|f\|_{2,\rho}}$

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- $\mathcal{T}$  verifies a (tight) log-Sobolev inequality with constant  $c$  if

$$H(f) \leq c \mathcal{E}(f, f) \quad \text{for all positive } f \quad (LSI(c))$$

*Gross 1975,*

*Bakry and Emery 1985,*

*Diaconis and Saloff-Coste 1996, Chen and Sheu 2003*

*Carlen Lieb 1993, Biane 1997, Bożejko 1999, Królak 2005*

- $\mathcal{T}$  is hypercontractive if

$$\|\mathcal{T}_t(f)\|_{q(t)} \leq \exp(d(1 - 2/q(t))) \|f\|_2$$

with  $d \geq 0$ ,  $q(t) = 1 + e^{2t/c}$  for some  $c > 0$   
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**Theorem** (Olkiewicz and Zegarlinski, JFA 1999)

(a)  $\mathcal{T}$  hypercontractive  $\Rightarrow \mathcal{T}$  verifies a weak LSI:

$$H(f) \leq c\mathcal{E}(f, f) + d\|f\|_{2,\rho}^2 \quad \forall f > 0$$

(b) if  $\mathcal{T}$  verifies a weak LSI and  $\mathcal{E}$  is “ $d_0$ -regular”

$\Rightarrow \mathcal{T}$  is hypercontractive with constant  $d$  replaced by  $d + cd_0$ .

Examples:

- quantum Ornstein-Uhlenbeck (RC&E.Sasso, PTRF 2008),
- diagonal preserving QMSs on  $M_2(\mathbb{C})$  (RC, IDAQP 2004)

## Relative Entropy

$$E(f) = \operatorname{tr}(\rho^{1/2} f \rho^{1/2} (\lg(\rho^{1/2} f \rho^{1/2}) - \lg \rho)) - \|f\|_1 \lg \|f\|_1, \quad f > 0$$

We use the embedding ( $f > 0$ )

$$I_{p,q}(f) = \rho^{-1/2p} (\rho^{1/2q} f \rho^{1/2q})^{q/p} \rho^{-1/2p}, \quad I_{p,q} : L^q \rightarrow L^p$$

and an associated operator valued relative entropy  $T_q$ ,

$$T_q(f) = (f \rho^{1/2q}) \lg(\rho^{1/2q} f \rho^{1/2q}) \rho^{-1/2q} - \frac{1}{2q} \{f, \lg \rho\} = -q \frac{d}{ds} I_{q+s,q}(f) \Big|_{s=0}$$

$$H(f) = \langle f, T_2(f) \rangle - \|f\|_2^2 \lg \|f\|_2$$

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- when  $\rho$  and  $f$  commute,

$$I_{p,q}(f) = f^{q/p}, \quad T_q(f) = f \lg f \quad \text{for any } p \text{ and } q$$

$$E(f) = \operatorname{tr}(\rho f \lg(f/\|f\|_1)), \quad H(f) = \frac{1}{2} E(f^2)$$

## Relative Entropy

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$$\nu = \frac{\rho^{1/2} f \rho^{1/2}}{\|f\|_1} \quad \text{if } [\rho, f] = 0 \quad \frac{f}{\|f\|_1} = \frac{\nu}{\rho}$$

$\nu$  is a state!

$$E(f) = \|f\|_1 \operatorname{tr}(\nu (\lg \nu - \lg \rho)) = \|f\|_1 S(\nu | \rho)$$

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## Entropy exponential decay

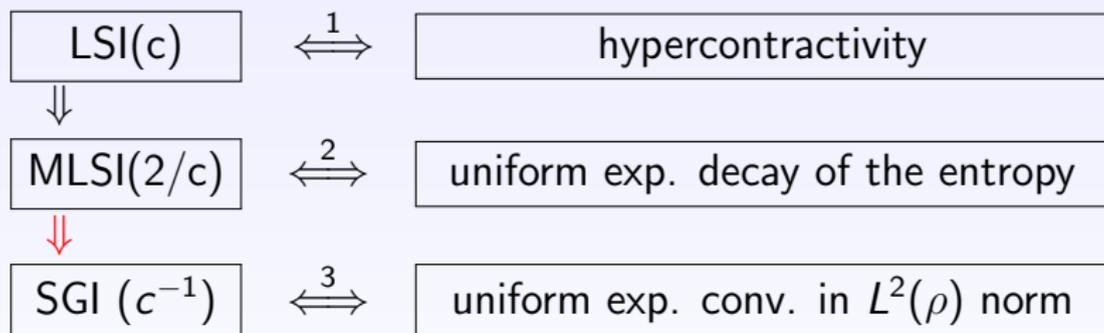
$\mathcal{T}$  verifies a **Modified Logarithmic Sobolev inequality** of constant  $c$  when

$$cE(f) \leq \mathcal{E}(\lg(\rho^{1/2}f\rho^{1/2}) - \lg \rho, f) \quad \forall f > 0 \quad (MLSI(c))$$

**Th.** The following conditions are equivalent:

- $E(\mathcal{T}_t f) \leq e^{-ct} E(f)$  for all  $f > 0$ ;
- MLSI( $c$ ).

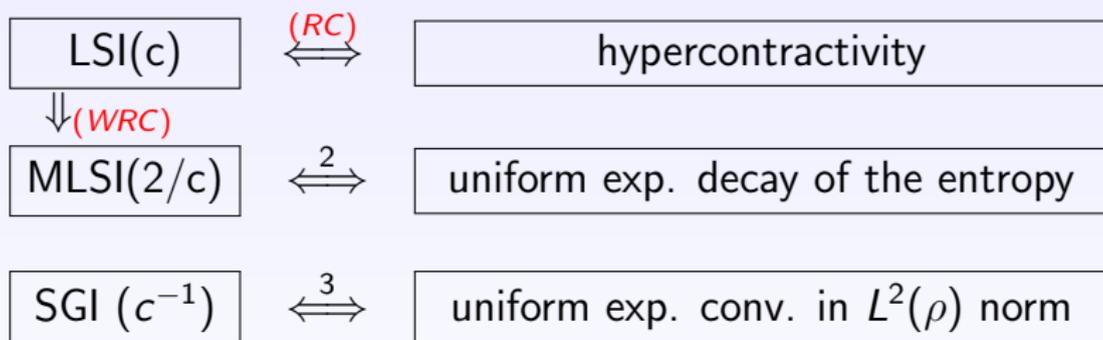
## Classical (commutative)



*Figure:* Scheme of hierarchical order in the commutative case

Bobkov&Tetali 2004, Bakry&Emery 1985, Guionnet&Zegarlinski 2003

## Quantum (non-commutative)



*Figure:* Scheme of hierarchical order in the commutative case

$$\text{LSI}(c) \Rightarrow \text{SGI}(c^{-1})$$

## *Regularity conditions*

For  $f$  positive and  $p$  conjugate to  $q$ ,

$$\mathcal{E}_q(f, f) := -\langle l_{p,q}(f), \mathcal{L}f \rangle = -\langle \rho^{-1/2p}(\rho^{1/2q} f \rho^{1/2q})^{q/p} \rho^{-1/2p}, \mathcal{L}f \rangle,$$

- $\mathcal{T}$  verifies the regularity condition (not tight if  $d_0 \neq 0$ ) when

$$(RC) \quad \mathcal{E}_2(l_{2,q}f, l_{2,q}f) \leq \frac{q^2}{4(q-1)} \mathcal{E}_q(f, f) + d_0 \|f\|_2^2 \quad \text{for all } q > 1$$

- $\mathcal{T}$  verifies the weak regularity condition with constant  $\beta$  when

$$(WRC-\beta) \quad \beta \mathcal{E}(l_{2,1}f, l_{2,1}f) \leq \mathcal{E}(\lg(\rho^{1/2} f \rho^{1/2}) - \lg \rho, f)$$

the (WRC) standard when the coefficient is  $\beta = 4$ .

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