

On stochastic acceleration

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The problem

The model: Particle in random force field: “Langevin without friction”

$$\ddot{y}(\tau) = G(y(\tau), \tau), \quad y(0) = y_0 \in \mathbb{R}^d, \quad \dot{y}(0) = v_0 \in \mathbb{R}^d$$

$$G(y, \tau) = - \sum_N \lambda_N \nabla_y W(y - x_N, \tau + \phi_N).$$

- $W \in C_0^\infty(B(0, \frac{1}{2}) \subset \mathbb{R}^d)$, rotationally invariant; $W(\tau + 1) = W(\tau)$, $\partial_\tau W(y, \tau) \neq 0$.

- The coupling constants λ_N and phases ϕ_N are bounded iid random variables.

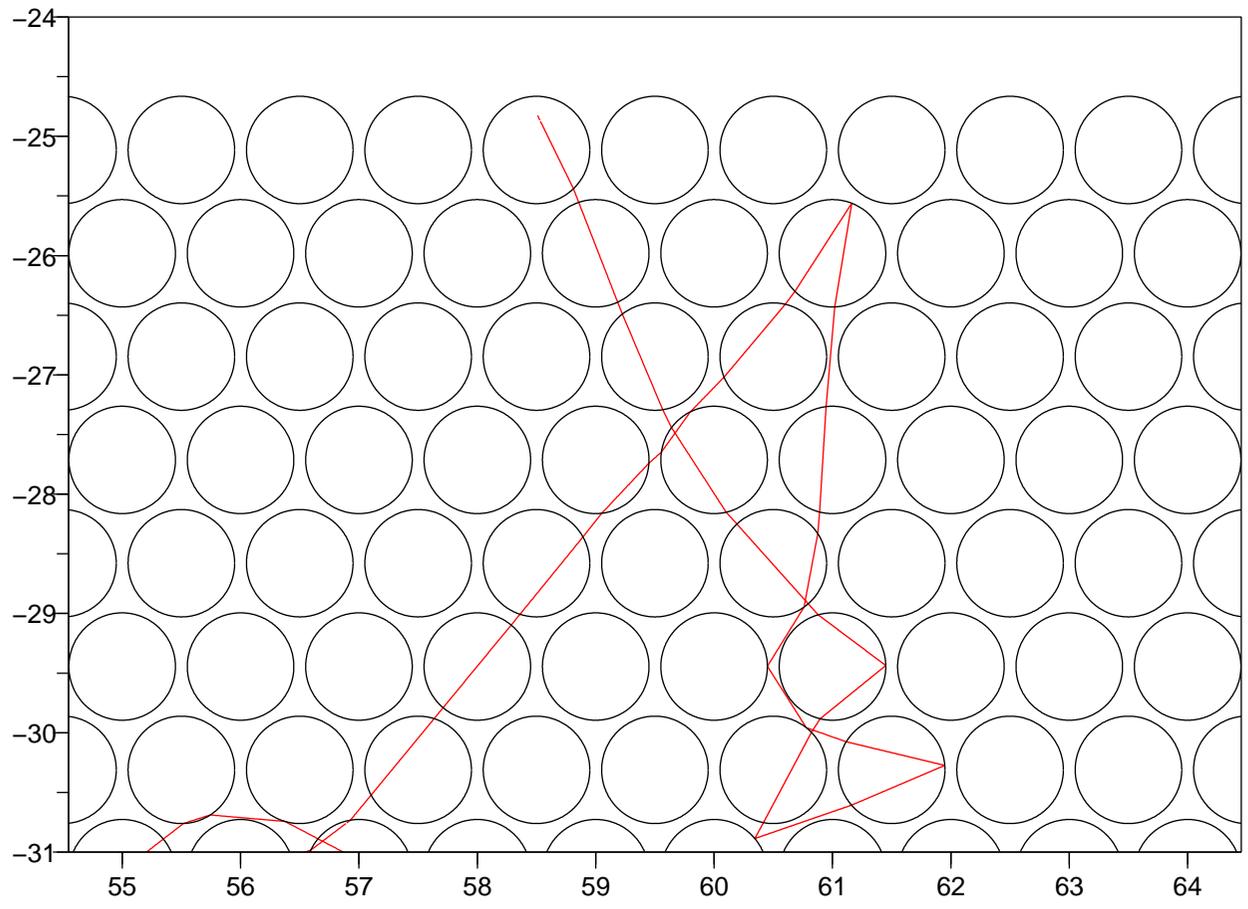
The ϕ_N are uniformly distributed in $[0, 1]$.

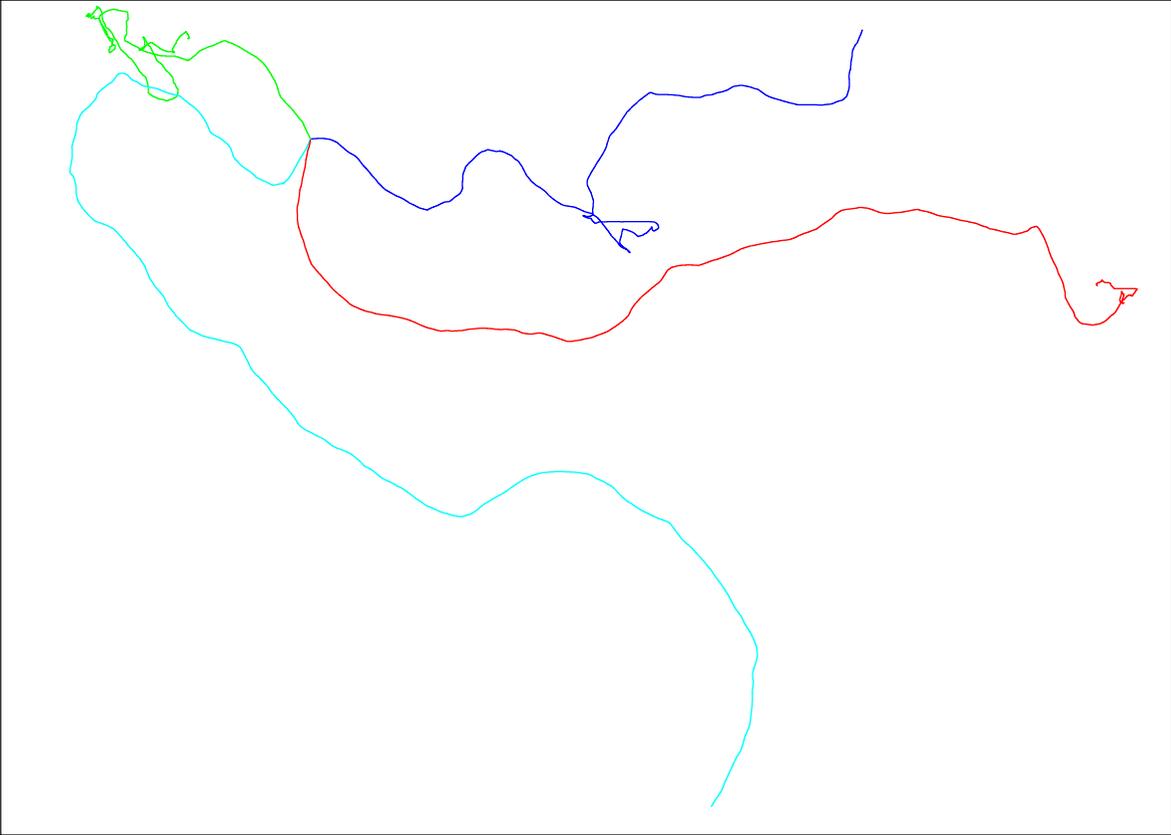
- The scattering centers x_N : a locally finite, possibly random family; finite horizon.

The question: Prove $v^2(\tau) \sim \tau^\alpha$, $y^2(\tau) \sim \tau^\beta$, in a suitable probabilistic sense, for some $\alpha, \beta \geq 0$, as $\tau \rightarrow +\infty$. How do the exponents depend on W , on the geometry of the (x_N) and on the dimension d ?

Note: If $\partial_\tau W(y, \tau) = 0$, then trivially $\alpha = 0$. But β is non-trivial. Keywords: (soft) Lorentz gas, Sinai billiard.

In this talk: Proof that $\alpha = 2/5$ for $d \geq 6$ and high initial v_0 , in a simplified model that ignores possible recollisions, that is described below.



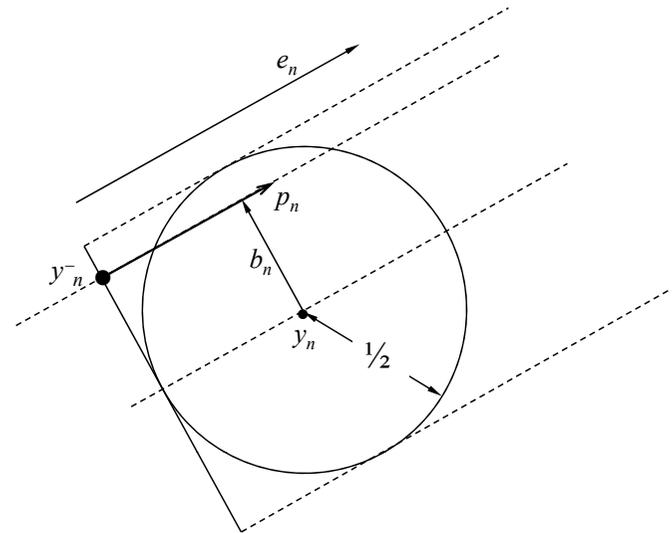


Describing the particle's motion

Think of $(y(\tau), v(\tau))$ as a stochastic process on the probability space generated by the (λ_N, ϕ_N, x_N) .

A trajectory = periods of free motion + scattering events at instants τ_n , centers $y_n := x_{N_n}$, where the particle is deviated by the local potential, with coupling constant $\lambda_n := \lambda_{N_n}$ and phase $\phi_n := \phi_{N_n} + \tau_n$; $v(\tau_n) = v_n = \|v_n\|e_n$ is the velocity just before the n th scattering event, at $y(\tau_n) = y_n^- = x_{N_n} - \frac{1}{2}e_n + b_n$, where $b_n \cdot e_n = 0$, so that b_n is the impact parameter (a vector!).

Repeated interaction: the process $(v_n, b_n, x_{N_n}, \tau_n)$ encodes (almost) all information about the full process $(y(\tau), v(\tau))$.



Determining $(v_n, b_n, \phi_n, \tau_n)$: dynamics and geometry

- Computing v_{n+1} : The velocity just after the n th scattering event is

$$v_{n+1} = v_n + R(v_n, \kappa_n),$$

where $\kappa_n = (b_n, \lambda_n, \phi_n)$ and for all v, b with $v \cdot b = 0$, the impulse function is

$$R(v, \kappa) = R(v, b, \lambda, \phi) = -\lambda \int_0^{+\infty} d\tau \nabla W(y(\tau), \tau + \phi)$$

in which $y(\tau)$ is the unique solution of

$$\ddot{y}(\tau) = -\lambda \nabla W(y(\tau), \tau + \phi), \quad y(0) = b - \frac{1}{2} \frac{v}{\|v\|}, \quad \dot{y}(0) = v.$$

This is where the **dynamics** takes place for v_n .

- Computing $b_{n+1}, \tau_{n+1}, \phi_{n+1}$ involves the **geometry** of the x_N .

Simplification: Trivialize the geometry and conserve only the non-trivial dynamics of the velocity proves by considering the following simplified model.

A Markov chain description

With the map $R(v, \kappa)$ defined on the previous slide, and considering a family $\kappa_n = (b_n, \lambda_n, \phi_n)$ of iid random variables, we can define the Markov chain

$$v_{n+1} = v_n + R(v_n, \kappa_n)$$

$$\tau_{n+1} = \tau_n + \frac{\ell}{\|v_{n+1}\|}$$

$$y_{n+1} = y_n + \ell e_{n+1}, \quad e_{n+1} = \frac{v_n}{\|v_n\|},$$

which provides a simplified model of the original problem. Here the λ_n are distributed in $[-1, 1]$, the ϕ_n are uniformly distributed in $[0, 1]$, and the b_n are uniformly distributed in the ball of radius $1/2$ centered at 0 and perpendicular to v_n . We will write $d\rho(\kappa_n)$ for the corresponding probability measure.

Remarks: (i) The first equation drives the two others, but is independent of them.

(ii) For fixed v , $R(v, \kappa)$ is only defined almost surely in κ . Trapping possible at low kinetic energy! This does not occur at high enough energy.

Goal: Prove $\|v_n\| \sim n^{1/6} \Rightarrow \tau_n \sim n^{5/6} \Rightarrow \|v(\tau_n)\| \sim \tau_n^{1/5}: \alpha = \frac{2}{5}$

Statement of the result

THEOREM 1 Suppose $d \geq 6$. Then

(i) For all $0 < p \leq 1$, for all $\nu > 0$, there exists $v_* > 0$ such that for all $\|v_0\| \geq v_*$, we have

$$\mathbb{P} \left(\forall t \geq 0, \left(\|v_0\| + t^{\frac{1}{5}} \right)^{1-\nu} \leq \|v(t)\| \leq \left(\|v_0\| + t^{\frac{1}{5}} \right)^{1+\nu} \right) \geq 1 - p,$$

(ii) For all $\nu > 0$, we have

$$\lim_{\|v_0\| \rightarrow +\infty} \mathbb{P} \left(\forall t > 0, \left(\|v_0\| + t^{\frac{1}{5}} \right)^{1-\nu} \leq \|v(t)\| \leq \left(\|v_0\| + t^{\frac{1}{5}} \right)^{1+\nu} \right) = 1.$$

QUESTIONS

- (i) What is the role of the high initial velocity condition?
- (ii) Why is d large needed?
- (iii) Is the result false in low dimension or if the initial velocity is low?
- (iv) What about moments of $\|v(t)\|$?
- (v) Can't you get rid of ν ?

Strategy of the proof

Step 1. Control the high velocity behaviour of $R(v, \kappa)$.

Step 2. Consider the family of Markov chains for $\Lambda_n^\epsilon \simeq \|v_n\|^3 / \|v_0\|^3$ indexed by $\epsilon = \|v_0\|^{-3}$ and show (using Step 1) their limit as $\epsilon \rightarrow 0$ is, under appropriate scaling (homogenisation or averaging), a Bessel process of dimension $\delta = \frac{d}{3} + \frac{1}{3}$.

Step 3. Introduce an auxiliary process $\eta_\ell \in \mathbb{Z}$ and corresponding stopping times t_ℓ such that (approximately) $\|v_{t_\ell}\|^3 \simeq 2^{\eta_\ell}$. Note that $\eta_{\ell+1} = \eta_\ell \pm 1$ and that $\Delta t_\ell = t_{\ell+1} - t_\ell$ is the time needed for the process $\|v_n\|^3$ to double or half its value.

Step 4. Exploit Step 2, properties of the Bessel process and the Portmanteau Lemma to show that, provided $d \geq 6$ and η_0 is large enough, η_ℓ is a submartingale and to control the Δt_ℓ . Basically: there exists $\mu > 0$ so that

$$\eta_\ell \sim \eta_0 + \mu\ell, \quad \Delta t_\ell \sim 2^{2\eta_\ell}$$

Step 5. Conclude.

Step 1. Extracted from Aguer, DB, Lafitte, Parris 2010.

Step 2-4. Adapted from Dolgopyat-Koralov 2009.

Step 1. High $\|v\|$ expansions

$$R(v, \kappa) = \sum_{k=1}^K \frac{\alpha^{(k)}(e, \kappa)}{\|v\|^k} + \mathcal{O}(\|v\|^{-K-1}), \quad e = \frac{v}{\|v\|},$$

with $\alpha^{(1)}(e, \kappa) = -\lambda \int_{-\infty}^{+\infty} dx \nabla W(b + xe, \phi)$. Then

$$\Delta E(v, \kappa) = \frac{1}{2} ((v + R(v, \kappa))^2 - v^2) = \sum_{\ell=0}^L \frac{\beta^{(\ell)}(e, \kappa)}{\|v\|^\ell} + \mathcal{O}(\|v\|^{-L-1}),$$

and

$$\beta^{(0)}(e, \kappa) = e \cdot \alpha^{(1)}(e, \kappa) = \mathbf{0}$$

$$\beta^{(1)}(e, \kappa) = e \cdot \alpha^{(2)}(e, \kappa) = \lambda \int_{-\infty}^{+\infty} dx \partial_\phi W(b + xe, \phi)$$

So, $\Delta E(v, \kappa)$ is of order $\|v\|^{-1}$ in this situation. We need the following information:

Define: $\overline{f(v)} = \int d\rho(b, \lambda, \phi) f(v, \kappa), \kappa = (b, \lambda, \phi)$.

THEOREM 2

$$\overline{\Delta E(v)} = \frac{B}{\|v\|^4} + O(\|v\|^{-5}), \quad \overline{(\Delta E(v))^2} = \frac{D^2}{\|v\|^2} + O(\|v\|^{-3}),$$

where $B = (d - 3)D^2/2$. In particular, for all unit vector $e \in \mathbb{R}^m$ and for $\ell = 0, 1, 2, 3$,

$$\overline{\beta^{(\ell)}(e)} = 0, \quad B = \overline{\beta^{(4)}(e)} \quad \text{and} \quad D^2 = \overline{(\beta^{(1)}(e))^2} > 0.$$

Conclusion: the energy transfer in a single scattering event is a random variable with fluctuations of order $\|v\|^{-1}$ and a mean of order $\|v\|^{-4}$. Then, with

$$\xi_n = \frac{\|v_n\|^3}{3D}, \quad \omega_n = \frac{\beta_n^{(1)}}{D} \quad \text{and} \quad \gamma = \frac{1}{3} \left(\frac{B}{D^2} + \frac{1}{2} \right) = \frac{1}{6}(d - 2) \geq -\frac{1}{6},$$

$$\xi_{n+1} = \xi_n + \omega_n + \frac{\gamma}{\xi_n} + O_0(\xi_n^{-1/3}) + O(\xi_n^{-4/3}).$$

Here the notation $O_0(\|v_n\|^{-1})$ means the term is $O(\|v_n\|^{-1})$ and of zero average.

WARNING Step 1 gives no information on the behaviour of $R(v, \kappa)$ when v_n and hence ξ_n is small. Obtaining such information is difficult because of the possibility of trapping. As a result, when looking at the Markov chain for $\xi_n \in \mathbb{R}_*^+$,

$$\xi_{n+1} = \xi_n + \omega_n + \frac{\gamma}{\xi_n} + O_0(\xi_n^{-1/3}) + O(\xi_n^{-4/3}),$$

it is a priori quite possible for ξ_n to become small, even if ξ_0 is large. This WILL happen if $\gamma = 0$, for example.

We will see that, with high probability, this does NOT happen if $\gamma > 1/2 \Leftrightarrow d \geq 6$.

This explains the origin of the latter condition in the central result.

The goal of **Steps 2 to 4** is to prove the following result on the Markov chain ξ_n .

$$\xi_{n+1} = \xi_n + \omega_n + \frac{\gamma}{\xi_n} + O_0(\xi_n^{-1/3}) + O(\xi_n^{-4/3}); \quad \mathbb{E}(\omega_n) = 0, \mathbb{E}(\omega_n^2) = 1, \omega_n \in [-M, M]$$

Note that all model-dependence is gone and that $\gamma = \frac{d-2}{6}$.

THEOREM 3 *Suppose $\gamma > \frac{1}{2}$. Then*

(i) *For all $0 < p \leq 1$, for all $\nu > 0$, there exists $\xi_* > \xi_+$ such that for all $\xi_0 \geq \xi_*$, we have*

$$\mathbb{P} \left(\forall k \in \mathbb{N}, \left(\xi_0 + k^{\frac{1}{2}} \right)^{1-\nu} \leq \xi_k \leq \left(\xi_0 + k^{\frac{1}{2}} \right)^{1+\nu} \right) \geq 1 - p, \quad (1)$$

(ii) *For all $\nu > 0$, we have*

$$\lim_{\xi_0 \rightarrow +\infty} \mathbb{P} \left(\forall n \in \mathbb{N}, \left(\xi_0 + n^{\frac{1}{2}} \right)^{1-\nu} \leq \xi_n \leq \left(\xi_0 + n^{\frac{1}{2}} \right)^{1+\nu} \right) = 1. \quad (2)$$

REMARK. Roughly, the theorem says $\xi_n \sim \sqrt{n}$, so $\|v_n\| \sim n^{1/6}$, which is the desired result.

Step 2. Homogenisation

Introduce $\varepsilon = \xi_0^{-1}$ and $\Lambda_n^\varepsilon := \varepsilon \xi_n$. Note that $\Lambda_0^\varepsilon = 1$, independently of ε .

Consider, for $n \in \mathbb{N}$, $t_n = n\varepsilon^2$. Define

$$\Lambda^\varepsilon(t) = \frac{t_{n+1} - t_n}{\varepsilon^2} \Lambda_n^\varepsilon + \frac{t - t_n}{\varepsilon^2} \Lambda_{n+1}^\varepsilon, \quad \text{for } t \in [t_n, t_{n+1}]. \quad (3)$$

THEOREM 4 Fixed $T \in \mathbb{R}_*^+$. If $\gamma \geq 1/2$, the processes $(\Lambda_t^\varepsilon)_{t \in [0, T]}$ converge, as $\varepsilon \rightarrow 0$, weakly to the Bessel process Λ_t , of dimension $2\gamma + 1$, and with initial condition 1:

$$d\Lambda_t = dB_t + \frac{\gamma}{\Lambda_t} dt, \quad \Lambda_0 = 1.$$

LEMMA Let $\gamma \geq 0$, and let Λ be a Bessel process of dimension $2\gamma + 1$ with $\Lambda(0) = 1$. Let

$$T_* = \inf\{t \geq 0 \mid \Lambda(t) \notin]\frac{1}{2}, 2[\}, \quad T_- = \inf\{t \geq 0 \mid \Lambda(t) < \frac{1}{2}\}, \quad T_+ = \inf\{t \geq 0 \mid \Lambda(t) > 2\}$$

(i) Then, for all $T \geq 0$, $0 < \mathbb{P}(T_* > T) < 1$.

(ii) If in addition $\gamma > \frac{1}{2}$, $\mathbb{P}(T_- > T_+) = \frac{2^{2\gamma-1} - 1}{2^{2\gamma-1} - 2^{1-2\gamma}} > \frac{1}{2}$

Step 3. The auxiliary process η_ℓ

Let $0 < M < L$ and define the intervals $J_\eta = [2^\eta - L; 2^\eta + L]$, for $\eta \in \mathbb{N}$. For $\eta \neq \eta'$ large enough, we have $J_\eta \cap J_{\eta'} = \emptyset$. Also, for η large enough, the process ξ_n cannot jump across one of these intervals without visiting it. Define η_0, τ_0 by

$$\xi_0 = 2^{\eta_0}, \quad \tau_0 = 0.$$

Then

$$\tau_1 := \inf\{k > \tau_0 \mid \xi_k \in J_{\eta_0-1} \cup J_{\eta_0+1}\}. \quad (4)$$

We define $\eta_1 = \eta_0 + 1$, if $\xi_{\tau_1} \in J_{\eta_0+1}$, and $\eta_1 = \eta_0 - 1$, if $\xi_{\tau_1} \in J_{\eta_0-1}$.

We then proceed recursively:

$$\tau_{\ell+1} = \inf\{k > \tau_\ell \mid \xi_k \in J_{\eta_\ell-1} \cup J_{\eta_\ell+1}\},$$

$$\eta_{\ell+1} = \eta_\ell + 1, \text{ if } \xi_{\tau_{\ell+1}} \in J_{\eta_\ell+1}, \text{ and } \eta_{\ell+1} = \eta_\ell - 1, \text{ if } \xi_{\tau_{\ell+1}} \in J_{\eta_\ell-1}.$$

We want to show that, for some $\mu > 0$,

$$\eta_\ell \sim \eta_0 + \mu\ell, \quad \tau_\ell \sim 2^{2\mu\ell}, \quad \text{and hence } \xi_{\tau_\ell} \sim \sqrt{\tau_\ell}.$$

Step 4. Controlling η_ℓ (and Δt_ℓ)

The process η_ℓ is not a Markov process. To control its asymptotic behaviour, we show it is, with high probability, a submartingale, if η_0 is sufficiently large, and control its jump probabilities $\mathbb{P}(\eta_{\ell+1} = \eta_\ell \pm 1 \mid \eta_\ell, \dots, \eta_0)$.

The transience of the chain ξ_k is essential in the arguments of this section; it is, as we shall see, ensured by the condition that $\gamma > \frac{1}{2}$.

PROPOSITION

- (i) *Suppose $\gamma > \frac{1}{2}$. For all $\delta > 0$ there exists $\tilde{\eta} > \eta_+$ such that for all $\ell \in \mathbb{N}^*$ and for almost all $\eta_0, \eta_1, \dots, \eta_{\ell-1} \geq \eta_+, \eta_\ell > \tilde{\eta}$, we have*

$$|\mathbb{P}(\eta_{\ell+1} = \eta_\ell \pm 1 \mid \eta_\ell, \dots, \eta_0) - p_\pm| < \delta,$$

where $p_+ = \frac{2^{2\gamma-1}-1}{2^{2\gamma-1}-2^{1-2\gamma}} > \frac{1}{2}$ and $p_- = 1 - p_+$.

- (ii) *For all $0 < p \leq 1$ and for all $\delta > 0$, there exists $\eta_* > \eta_+$ so that for all $\eta_0 \geq \eta_*$*

$$\mathbb{P}(|\eta_\ell - \mu\ell - \eta_0| \leq \delta(\ell + \eta_0), \forall \ell \in \mathbb{N}) \geq 1 - p,$$

where $\mu = 2p_+ - 1 > 0$.