Entanglement of Bipartite Quantum Systems driven by Repeated Interactions

Julien Deschamps

joint work with Stéphane Attal and Clément Pellegrini

Autrans, July 2013

Plan of the Talk

1. Description of the Bipartite Model
2. Continuous Time Limit
3. Entanglement in the Spontaneous Emission
4. Thermal Environment

Description of the Bipartite Model

- Bipartite system is $\mathcal{H}_S = \mathcal{H}_S^A \otimes \mathcal{H}_S^B$ (not interacting together, isolated systems)
- Free Hamiltonian of $\mathcal{H}_S$ is

$$H_S = H^A \otimes I + I \otimes H^B,$$

where H^A and H^B are the free Hamiltonian of $\mathcal{H}_S^A$ and $\mathcal{H}_S^B$

- Environment is a collection of identical and independent pieces represented by

$$T\Phi = \bigotimes_{n \in \mathbb{N}^*} \mathcal{K}_n ,$$

where $\mathcal{K}_n = \mathcal{K} = \mathbb{C}^{N+1}$.

- More precisely, the free evolution of each piece is given by a Hamiltonian H^R. Fix an orthonormal basis $\{e_0, ..., e_N\}$ of $\mathbb{C}^{N+1}$ which diagonalizes H^R (where the vector e_0 represents the ground state allowing us to define the countable tensor product). Note that

$$H^R = \sum_{j=0}^{N} \lambda_j \, |e_i\rangle\langle e_i| ,$$

where the λ_j's are real

Description of the Bipartite Model

- Bipartite system is $\mathcal{H}_S = \mathcal{H}_S^A \otimes \mathcal{H}_S^B$ (not interacting together, isolated systems)
- Free Hamiltonian of $\mathcal{H}_S$ is

$$H_S = H^A \otimes I + I \otimes H^B,$$

where H^A and H^B are the free Hamiltonian of $\mathcal{H}_S^A$ and $\mathcal{H}_S^B$

- Environment is a collection of identical and independent pieces represented by

$$T\Phi = \bigotimes_{n \in \mathbb{N}^*} \mathcal{K}_n \,,$$

where $\mathcal{K}_n = \mathcal{K} = \mathbb{C}^{N+1}$.

- More precisely, the free evolution of each piece is given by a Hamiltonian H^R. Fix an orthonormal basis $\{e_0, \ldots, e_N\}$ of $\mathbb{C}^{N+1}$ which diagonalizes H^R (where the vector e_0 represents the ground state allowing us to define the countable tensor product). Note that

$$H^R = \sum_{j=0}^{N} \lambda_j \, |e_i\rangle\langle e_i| \,,$$

where the λ_j's are real.

Interaction between $\mathcal{H}_S$ and $\mathcal{K}$

- Usual repeated interaction scheme for which each interaction is given by a Hamiltonian on $\mathcal{H}_S^A \otimes \mathcal{H}_S^B \otimes \mathcal{K}$

$$\tilde{H}_{tot} = H^A \otimes I \otimes I + I \otimes H^B \otimes I + I \otimes I \otimes H^R + \tilde{H}_I ,$$

inducing a unitary operator $\tilde{U} = e^{-ih\tilde{H}_{tot}}$.

- Bipartite repeated interaction scheme

First, interaction between $\mathcal{H}_S^A$ and $\mathcal{K}$ whose Hamiltonian is

$$H^A_{tot} = H^A \otimes I \otimes I + I \otimes I \otimes H^R + H_I^A$$

(with H_I^A trivial on $\mathcal{H}_S^B$) defining a unitary operator $U^A = e^{-ihH^A_{tot}}$.
Then, interaction between $\mathcal{H}_S^B$ and $\mathcal{K}$ whose Hamiltonian is

$$H^B_{tot} = I \otimes H^B \otimes I + I \otimes I \otimes H^R + H_I^B$$

(with H_I^B trivial on $\mathcal{H}_S^A$) defining a unitary operator $U^B = e^{-ihH^B_{tot}}$.

- Interaction between $\mathcal{H}_S$ et $\mathcal{K}$ given by $U = U^B U^A$.

Interaction between $\mathcal{H}_S$ and $\mathcal{K}$

- Usual repeated interaction scheme for which each interaction is given by a Hamiltonian on $\mathcal{H}_S^A \otimes \mathcal{H}_S^B \otimes \mathcal{K}$

$$\tilde{H}_{tot} = H^A \otimes I \otimes I + I \otimes H^B \otimes I + I \otimes I \otimes H^R + \tilde{H}_I \,,$$

inducing a unitary operator $\tilde{U} = e^{-ih\tilde{H}_{tot}}$.

- Bipartite repeated interaction scheme
First, interaction between $\mathcal{H}_S^A$ and $\mathcal{K}$ whose Hamiltonian is

$$H_{tot}^A = H^A \otimes I \otimes I + I \otimes I \otimes H^R + H_I^A$$

(with H_I^A trivial on $\mathcal{H}_S^B$) defining a unitary operator $U^A = e^{-ihH_{tot}^A}$.
Then, interaction between $\mathcal{H}_S^B$ and $\mathcal{K}$ whose Hamiltonian is

$$H_{tot}^B = I \otimes H^B \otimes I + I \otimes I \otimes H^R + H_I^B$$

(with H_I^B trivial on $\mathcal{H}_S^A$) defining a unitary operator $U^B = e^{-ihH_{tot}^B}$.
- Interaction between $\mathcal{H}_S$ et $\mathcal{K}$ given by $U = U^B U^A$.

Interaction between $\mathcal{H}_S$ and $\mathcal{K}$

- Canonical operators $a_j^i = |e_j\rangle\langle e_i|$ on $\mathbb{C}^{N+1}$
- Interaction Hamiltonians

$$H_I^A = \sum_{j=1}^{N} V_j \otimes I \otimes a_j^0 + V_j^* \otimes I \otimes a_0^j \,,$$

where the V_j's are operators on $\mathcal{H}_S^A$ and

$$H_I^B = \sum_{j=1}^{N} I \otimes W_j \otimes a_j^0 + I \otimes W_j^* \otimes a_0^j \,,$$

where the W_j's are operators on $\mathcal{H}_S^B$.

Interaction between $\mathcal{H}_S$ and $\mathcal{K}$

- Canonical operators $a_j^i = |e_j\rangle\langle e_i|$ on $\mathbb{C}^{N+1}$
- Interaction Hamiltonians

$$H_I^A = \sum_{j=1}^{N} V_j \otimes I \otimes a_j^0 + V_j^* \otimes I \otimes a_0^j \,,$$

where the V_j's are operators on $\mathcal{H}_S^A$ and

$$H_I^B = \sum_{j=1}^{N} I \otimes W_j \otimes a_j^0 + I \otimes W_j^* \otimes a_0^j \,,$$

where the W_j's are operators on $\mathcal{H}_S^B$.

Interaction between $\mathcal{H}_S$ and $T\Phi$

- Natural ampliation of U to $\mathcal{H}_S \otimes T\Phi$ and written with respect to the canonical basis

$$U_n = \sum_{ij} U^i_j \otimes a^i_j(n)\,.$$

- The whole evolution is then given by

$$V_n = U_n U_{n-1} \ldots U_1\,.$$

- Dynamics of the V_n's

$$\begin{aligned} V_{n+1} &= U_{n+1}\, V_n \\ &= \sum_{ij} U^i_j \otimes a^i_j(n+1) V_n \\ &= \sum_{ij} U^i_j\, V_n\, a^i_j(n+1) \end{aligned}$$

$$V_{n+1} - V_n = \sum_{i,j} (U^i_j - \delta_{ij}) V_n a^i_j(n+1)\,.$$

Continuous Time Limit

- Interaction between $\mathcal{K}_n$ and $\mathcal{H}_S^A$ represented by $U^A = e^{-ihH^A_{tot}}$ on $\mathcal{H}_S^A \otimes \mathcal{H}_S^B \otimes \mathcal{K}$ with

$$H^A_{tot} = H^A \otimes I \otimes I + I \otimes I \otimes H^R + \frac{1}{\sqrt{h}} \sum_{j=1}^{N} V_j \otimes I \otimes a_j^0 + V_j^* \otimes I \otimes a_0^j \,,$$

- Interaction between $\mathcal{K}_n$ and $\mathcal{H}_S^B$ represented by $U^B = e^{-ihH^B_{tot}}$ on $\mathcal{H}_S^A \otimes \mathcal{H}_S^B \otimes \mathcal{K}$ with

$$H^B_{tot} = I \otimes H^B \otimes I + I \otimes I \otimes H^R + \frac{1}{\sqrt{h}} \sum_{j=1}^{N} I \otimes W_j \otimes a_j^0 + I \otimes W_j^* \otimes a_0^j \,,$$

- Interaction between $\mathcal{H}_S$ and $\mathcal{K}$ given by $U = U^B U^A$.

Effective Interaction Hamiltonian

Theorem (Attal, D., Pellegrini)

When h goes to 0, the limit evolution is given by the unitary dynamics

$$dU_t = L_0^0 U_t dt + \sum_j L_0^j U_t da_0^j(t) + L_j^0 U_t da_j^0(t)\,,$$

where

$$L_0^0 = -i(H_0^{A,B} + 2\lambda_0 I \otimes I) - \frac{1}{2}\sum_j S_j^* S_j,$$

$$L_j^0 = -iS_j\,, \qquad L_0^j = -iS_j^*\,.$$

with $S_j = V_j \otimes I + I \otimes W_j$ and the Hamiltonian

$$H_0^{A,B} = H^A \otimes I + I \otimes H^B + \frac{i}{2}\sum_j V_j^* \otimes W_j - V_j \otimes W_j^*\,.$$

Remarks on the Effective Interaction Hamiltonian

$$H_c = \frac{i}{2}\sum_j V_j^* \otimes W_j - V_j \otimes W_j^*$$

- This evolution can be obtained from the usual repeated interaction scheme if one starts with interacting systems whose interaction is given H_c.
- H_c is not symmetric. One can read on it the order of interactions.
- Classical case where $V_j = V_j^*$ and $W_j = W_j^*$, there is no interaction.

Spontaneous Emission

- System : $\mathcal{H}_S^A = \mathcal{H}_S^B = \mathcal{K} = \mathbb{C}^2$.
- Hamiltonians : H^A, H^B et H^R are $\sigma_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
- Limit Evolution :

$$dU_t = \left[-i(\sigma_z \otimes I + I \otimes \sigma_z + 2I \otimes I) - \frac{1}{2} S^* S + \frac{1}{2}(a_1^0 \otimes a_0^1 - a_0^1 \otimes a_1^0) \right] U_t dt \\ - iSU_t da_1^0(t) - iS^* U_t da_0^1(t) ,$$

with $S = a_0^1 \otimes I + I \otimes a_0^1$ and whose associated Lindblad generator is

$$L(\rho) = -i \left[\sigma_z \otimes I + I \otimes \sigma_z + \frac{i}{2}(a_1^0 \otimes a_0^1 - a_0^1 \otimes a_1^0), \rho \right] \\ + \frac{1}{2}\Big(2S\rho S^* - S^* S\rho - \rho S^* S \Big) .$$

Entanglement of Formation for a Pure State

Theorem

A pure state $|\Psi\rangle$ on $\mathcal{H}_S^A \otimes \mathcal{H}_S^B$ can be written as

$$|\Psi\rangle = \sum_j d_j \, |\Psi_j^A\rangle \otimes |\Psi_j^B\rangle \,,$$

where the d_j's are positive, the $|\Psi_j^A\rangle$'s and the $|\Psi_j^B\rangle$'s form respectively an orthonormal basis of $\mathcal{H}_S^A$ and $\mathcal{H}_S^B$.

The entanglement of formation of $|\Psi\rangle$ is defined by the following entropy

$$E(|\Psi\rangle) = -\sum_j d_j^2 \log(d_j^2) \,.$$

Entanglement of Formation for a Mixed State

Theorem

A density matrix ρ on $\mathcal{H}_S^A \otimes \mathcal{H}_S^B$ can be written as

$$\rho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k| \, ,$$

where the p_k's are non-negative, the $|\Psi_k\rangle$'s form an orthonormal basis of $\mathcal{H}_S^A \otimes \mathcal{H}_S^B$.

The entanglement of formation of ρ is defined by

$$E(\rho) = \min \sum_k p_k \, E(|\Psi_k\rangle) \, ,$$

where the minimum is on all the decompositions of ρ.

Case of X-states

Definition

A state of $\mathcal{H}_S^A \otimes \mathcal{H}_S^B$ is a X-state in the basis $(|e_0 \otimes e_0\rangle, |e_0 \otimes e_1\rangle, |e_1 \otimes e_0\rangle, |e_1 \otimes e_1\rangle)$, if ρ is of the following form

$$\rho = \begin{pmatrix} a & 0 & 0 & y \\ 0 & b & x & 0 \\ 0 & \overline{x} & c & 0 \\ \overline{y} & 0 & 0 & d \end{pmatrix}$$

with the conditions a, b, c, d non-negative reals such that $a + b + c + d = 1$, $|y|^2 \leq ad$ and $|x|^2 \leq bc$.

Case of X-states

Theorem (Vogelsberger (1), Wooters (2))

If ρ is a X-state of the form

$$\rho = \begin{pmatrix} a & 0 & 0 & y \\ 0 & b & x & 0 \\ 0 & \overline{x} & c & 0 \\ \overline{y} & 0 & 0 & d \end{pmatrix},$$

the concurrence of Wooters is

$$C(\rho) = 2\max(0, |y| - \sqrt{bc}, |x| - \sqrt{ad}) \tag{1}$$

and the entanglement of formation is given by the formula

$$E(\rho) = h\left(\frac{1+\sqrt{1-C(\rho)}}{2}\right), \tag{2}$$

where $h(x) = -x\log(x) - (1-x)\log(1-x)$.

Entanglement in the Spontaneous Emission

- The state $|e_0 \otimes e_0\rangle\langle e_0 \otimes e_0|$ is invariant.
- Initial state $|e_0 \otimes e_1\rangle\langle e_0 \otimes e_1|$
- Evolution is for all $t > 0$

$$\rho_t^{01} = e^{tL}(|e_0 \otimes e_1\rangle\langle e_0 \otimes e_1|) = \begin{pmatrix} 1 - e^{-t} & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- Entanglement of formation

$$E(\rho_t^{01}) = 0 \,.$$

Entanglement in the Spontaneous Emission

- The state $|e_0 \otimes e_0\rangle\langle e_0 \otimes e_0|$ is invariant.

- Initial state $|e_0 \otimes e_1\rangle\langle e_0 \otimes e_1|$
- Evolution is for all $t > 0$

$$\rho_t^{01} = e^{tL}(|e_0 \otimes e_1\rangle\langle e_0 \otimes e_1|) = \begin{pmatrix} 1 - e^{-t} & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- Entanglement of formation

$$E(\rho_t^{01}) = 0\,.$$

Entanglement in the Spontaneous Emission

- Initial state $|e_1 \otimes e_0\rangle\langle e_1 \otimes e_0|$
- Evolution for all $t > 0$

$$\rho_t^{10} = e^{tL}(|e_1 \otimes e_0\rangle\langle e_1 \otimes e_0|) = \begin{pmatrix} 1-(1+t^2)e^{-t} & 0 & 0 & 0 \\ 0 & e^{-t} & -te^{-t} & 0 \\ 0 & -te^{-t} & t^2e^{-t} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

- The entanglement of formation is

$$E(\rho_t^{10}) = h\left(\frac{1+\sqrt{1-4t^2e^{-2t}}}{2}\right),$$

where $h(x) = -x\log(x) - (1-x)\log(1-x)$.

Entanglement in the Spontaneous Emission

- Initial state $|e_1 \otimes e_1\rangle\langle e_1 \otimes e_1|$
- Evolution for all $t > 0$

$$\rho_t^{11} =$$

$$\begin{pmatrix} 1-(t^2-4t+6)e^{-t}+5e^{-2t} & 0 & 0 & 0 \\ 0 & (t^2-4t+5)e^{-t}-5e^{-2t} & (2-t)e^{-t}-2te^{-2t} & 0 \\ 0 & (2-t)e^{-t}-2e^{-2t} & e^{-t}-e^{-2t} & 0 \\ 0 & 0 & 0 & e^{-2t} \end{pmatrix}.$$

- The concurrence of Wooters is

$$C(\rho_t^{11}) = 2\max[0, |(2-t)e^{-t}-2e^{-2t}| - \sqrt{(1-(t^2-4t+6)e^{-t}+5e^{-2t})e^{-2t}}].$$

- The entanglement of formation is

$$E(\rho_t^{11}) = h\left(\frac{1+\sqrt{1-C(\rho_t^{11})}}{2}\right).$$

Environment : a Gibbs state for a positive β

$$\omega_\beta = \frac{1}{Z} e^{-\beta H^R} = \sum_j \beta_j |e_j\rangle\langle e_j| \,,$$

where Z is a normalizing constant, $\{e_0, \ldots, e_N\}$ an eigen-basis and $\beta_j = e^{-\beta\lambda_j}/Z$ satisfying $\sum_j \beta_j = 1$.

Proposition

The Lindblad generator of the limit evolution is

$$\begin{aligned} L_\beta(\rho) = &- i \left[H^A \otimes I + I \otimes H^B + \frac{i}{2} \sum_{j=1}^{N} (\beta_j - \beta_0)(V_j \otimes W_j^* - V_j^* \otimes W_j), \rho \right] \\ &- \frac{1}{2} \sum_{j=1}^{N} \beta_j \big(S_j S_j^* \rho + \rho S_j S_j^* - 2 S_j^* \rho S_j \big) \\ &- \frac{1}{2} \sum_{j=1}^{N} \beta_0 \big(S_j^* S_j \rho + \rho S_j^* S_j - 2 S_j \rho S_j^* \big) \,, \end{aligned}$$

where $S_j = V_j \otimes I + I \otimes W_j$.

Physical Example

- Hilbert spaces $\mathcal{H}_S^A$, $\mathcal{H}_S^B$ and $\mathcal{K}$ are $\mathbb{C}^{N+1}$.
- We assume that the free evolutions satisfy $H^A = H^B = H^R$.
- The total Hamiltonian operators are

$$
\begin{aligned}
H_{tot}^A &= H^A \otimes I \otimes I + I \otimes I \otimes H^R + \frac{1}{\sqrt{h}} \sum_{j=1}^{N} a_j^0 \otimes I \otimes a_0^j + a_0^j \otimes I \otimes a_j^0 \,, \\
H_{tot}^B &= I \otimes H^B \otimes I + I \otimes I \otimes H^R + \frac{1}{\sqrt{h}} \sum_{j=1}^{N} I \otimes a_j^0 \otimes a_0^j + I \otimes a_0^j \otimes a_j^0 .
\end{aligned}
$$

where $V_j = W_j = a_0^j$.

Limit Lindblad Generator

The limit Lindblad generator is

$$L_\beta(\rho) = -i\left[H^A \otimes I + I \otimes H^B + \frac{i}{2}\sum_{j=1}^{N}(\beta_j - \beta_0)(a_0^j \otimes a_j^0 - a_j^0 \otimes a_0^j), \rho\right]$$
$$- \frac{1}{2}\sum_{j=1}^{N}\beta_j\left(S_j S_j^* \rho + \rho S_j S_j^* - 2S_j^* \rho S_j\right)$$
$$- \frac{1}{2}\sum_{j=1}^{N}\beta_0\left(S_j^* S_j \rho + \rho S_j^* S_j - 2S_j \rho S_j^*\right),$$

where $S_j = a_0^j \otimes I + I \otimes a_0^j$.

Return to Equilibrium

The system has the property of return to equilibrium if there exists an invariant state such that for any initial state ρ and all observable X

$$\lim_{t \to +\infty} \mathrm{Tr}(e^{tL_\beta}(\rho)X) = \mathrm{Tr}(\rho_\infty X) .$$

Proposition

On $\mathcal{H}_S^A \otimes \mathcal{H}_S^B$, the dynamical system has the property of return to equilibrium.

Moreover, the limit invariant state is

$$\rho_\beta = \frac{e^{-\beta(H^A \otimes I + I \otimes H^B)}}{Z} ,$$

where Z is a normalizing constant.

Thank You !