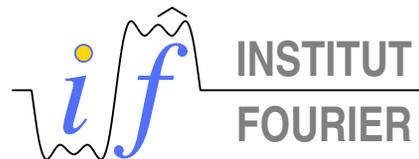


Transport Properties of Quantum Walks, Deterministic and Random

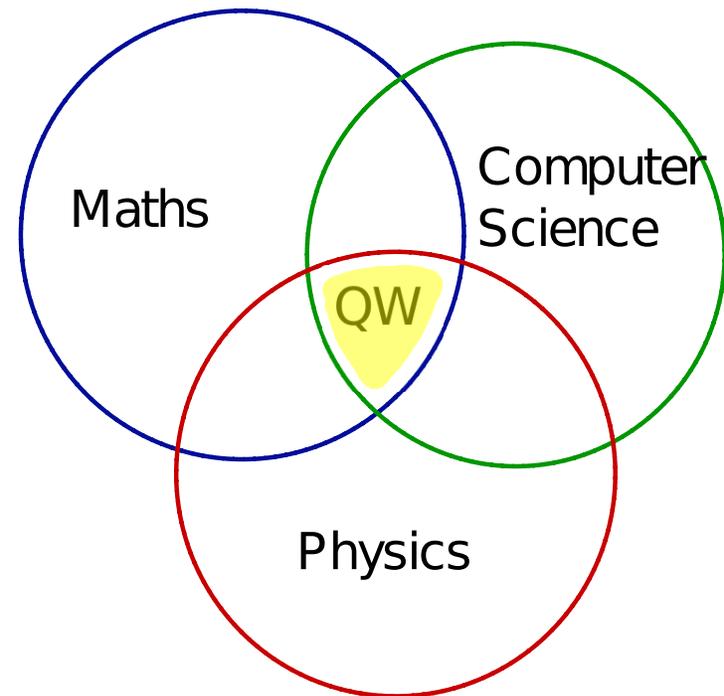
Alain JOYE



Unitary Quantum Walks

Interdisciplinary Concept

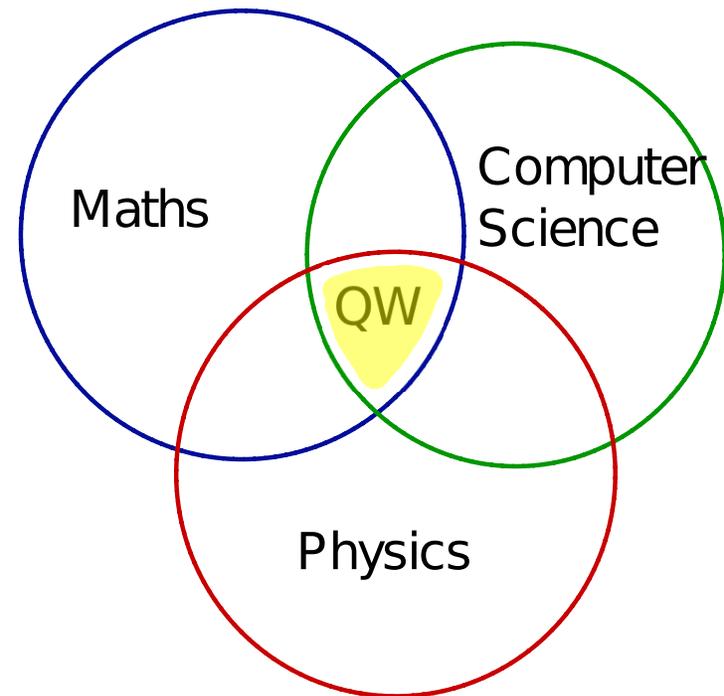
- Quantum from Quantum Physics
- Walk from Probability Theory
- Well studied in Computer Science



Unitary Quantum Walks

Interdisciplinary Concept

- **Quantum** from Quantum Physics
- **Walk** from Probability Theory
- Well studied in **Computer Science**



Interest for

- **Computer Scientists:** elaboration of softwares for Quantum Computers
Aharonov et al '93, Grover '96, Childs et al '03 ...
- **Physicists:** effective dynamics of complex Quantum Systems
Feynman '82, Chalker-Coddington '88, Meyer '96 ...
- **Mathematicians:** non-commutative extensions of Random Walks
Nayak et al '00, Ambainis et al 01, Konno '02, ...

Search Algorithm

Search in unstructured list

Q: Does $S_N = \{x_1, x_2, \dots, x_N\}$ contain the elmt. x ? If so, find it.

Algorithmic Answer:

pick x_1 $\left\{ \begin{array}{ll} \text{If } x_1 = x & \longrightarrow \text{stop} \\ \text{If } x_1 \neq x & \longrightarrow \text{pick } x_2 \quad \text{iterate} \end{array} \right.$

- Algorithm typically requires $O(N)$ steps to determine if $x \in S_N$ and find it.

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a) $x \notin S_N$ (with proba p). $\Rightarrow N$ steps required.

b) $x \in S_N$ (with proba $1 - p$). \Rightarrow Stop after $k \leq N$ steps.

$$\text{prob}(x \text{ found at step } k) = 1/N \Rightarrow \bar{T} = \sum_{k=1}^N \frac{k}{N} = \frac{N+1}{2}.$$

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- Assuming random choice of states

i.e. a random walk on $S_N \Rightarrow$ same conclusions.

Quantum Version: $S_N \rightsquigarrow \mathbb{C}^N$

Q: How to find the (normalized) state $x \in \mathbb{C}^N$?

Assume $x \in \mathbb{C}^N$ charact. by an oracle $U_x \in U(N)$ s.t.

$$U_x x = -x \text{ and } U_x \varphi = +\varphi, \text{ for all } \varphi \in \text{span}\{x\}^\perp$$

Algorithmic Answer:

Let $s = \frac{1}{\sqrt{N}} \sum_{j=1}^N |j\rangle$ and $U_s \in U(N)$ s.t.

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Sequence of States

$$s \rightsquigarrow s_1 = (U_s U_x)^1 s \rightsquigarrow s_2 = (U_s U_x)^2 s \rightsquigarrow \dots \rightsquigarrow s_r = (U_s U_x)^r s$$

Fact: For $r(N) \simeq \pi\sqrt{N}/4$, $|\langle x | s_{r(N)} \rangle|^2 \simeq 1 - O(1/N)$ (if $\langle x | s \rangle \neq 0$)

Quantum is Faster

It takes $O(\sqrt{N})$ steps to find the state x with large prob.

by a Quantum Walk in state space

Quantum Walk on \mathbb{Z}

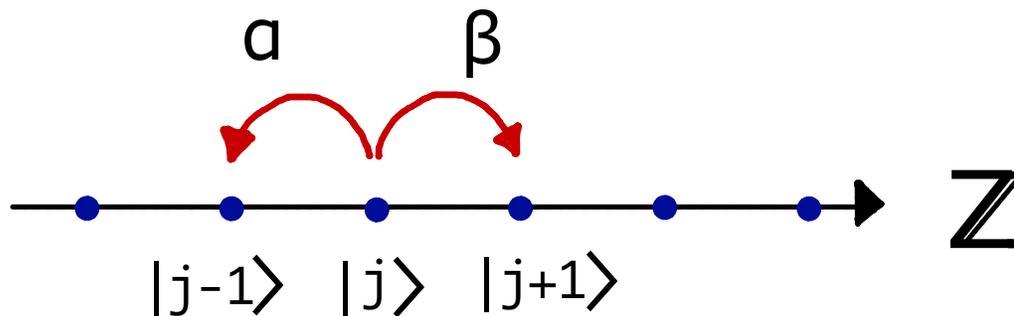
First Try

- Hilbert space $l^2(\mathbb{Z})$ with ONB $\{|j\rangle\}_{j \in \mathbb{Z}}$
- Unitary op. U on $l^2(\mathbb{Z})$
- Jumps from $|j\rangle$ to $|j+1\rangle$, $|j-1\rangle$ only, for all $j \in \mathbb{Z}$

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- Jumps from $|j\rangle$ to $|j+1\rangle$, $|j-1\rangle$ only, for all $j \in \mathbb{Z}$
- $U|j\rangle = \alpha|j-1\rangle + \beta|j+1\rangle$, $\alpha, \beta \in \mathbb{C}$ s.t. $|\alpha|^2 + |\beta|^2 = 1$

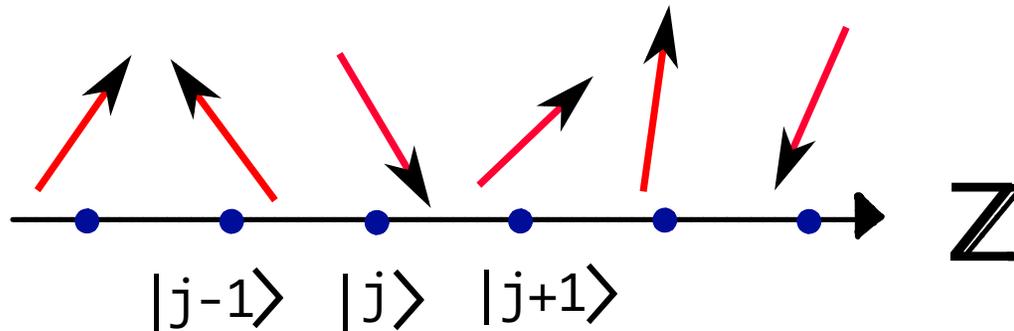


- Orthogonality: $0 = \langle j|j+2\rangle = \langle Uj|U(j+2)\rangle = \bar{\beta}\alpha$
- Only possibilities: Left ($\beta = 0$) or Right ($\alpha = 0$) **Shifts**

Quantum Walk on \mathbb{Z}

Second Try

- Add Internal degree of freedom: \mathbb{C}^2 a.k.a Spin



- Hilbert space $l^2(\mathbb{Z}; \mathbb{C}^2) \simeq l^2(\mathbb{Z}) \otimes \mathbb{C}^2 \ni \Psi = \sum_{j \in \mathbb{Z}} |j\rangle \otimes \psi(j)$, where

$$\psi(j) = \begin{pmatrix} a(j) \\ b(j) \end{pmatrix} = a(j)|+1\rangle + b(j)|-1\rangle \in \mathbb{C}^2,$$

$$\text{s.t. } \sum_{j \in \mathbb{Z}} \|\psi(j)\|^2 = \sum_{j \in \mathbb{Z}} |a(j)|^2 + |b(j)|^2 < \infty.$$

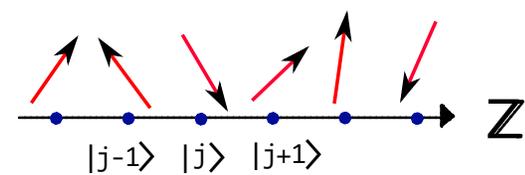
$$\rightsquigarrow \text{ONB } \{|j \otimes \tau\rangle\}_{j \in \mathbb{Z}}^{\tau \in \{+1, -1\}} = \{|j \otimes +1\rangle, |j \otimes -1\rangle\}_{j \in \mathbb{Z}}$$

- QM interpretation:

$$\text{prob. } (\Psi \text{ on "site" } j) = \|\psi(j)\|^2$$

(Ψ normalized)

Coined Quantum Walk I

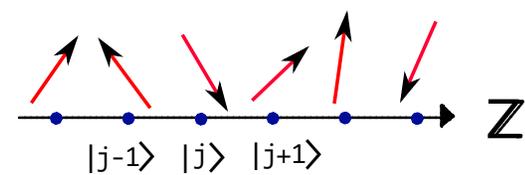


Construction of Unitary op. U on $l^2(\mathbb{Z}) \otimes \mathbb{C}^2$:

- Jumps from $|j \otimes \tau\rangle$ to $|(j + 1) \otimes \tau'\rangle$, $|(j - 1) \otimes \tau''\rangle$ only,
for all $j \in \mathbb{Z}, \tau, \tau', \tau'' \in \{+1, -1\}$
- **First Step:** Let $C \in U(2)$ to act **locally** on **Spin** deg. of freed.

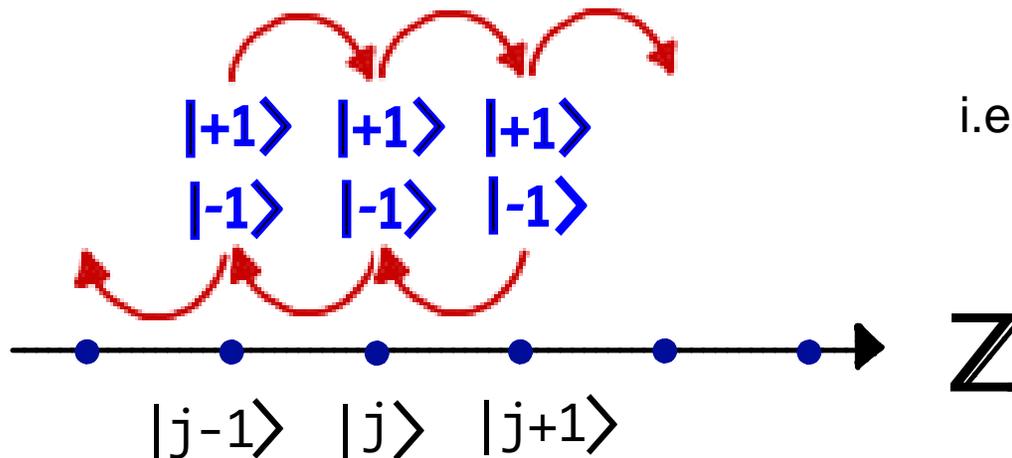
$$(\mathbb{I} \otimes C)|j\rangle \otimes \varphi = |j\rangle \otimes (C\varphi), \quad \forall \varphi \in \mathbb{C}^2$$

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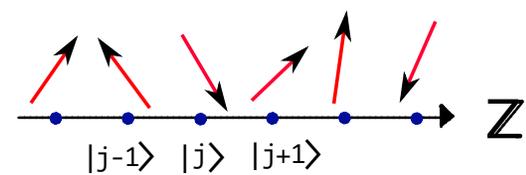
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 $(\mathbb{I} \otimes C)|j\rangle \otimes \varphi = |j\rangle \otimes (C\varphi)$, $\forall \varphi \in \mathbb{C}^2$
- **Second Step:** Spin-dependent shift op. $S \in \mathcal{B}(l^2(\mathbb{Z}) \otimes \mathbb{C}^2)$
 $S|j \otimes +1\rangle = |(j+1) \otimes +1\rangle$, $S|j \otimes -1\rangle = |(j-1) \otimes -1\rangle$



i.e. $S \simeq S_+ \oplus S_-$

Coined Quantum Walk II



Definition:

$$U(C) = S(\mathbb{I} \otimes C) \text{ on } l^2(\mathbb{Z}) \otimes \mathbb{C}^2$$

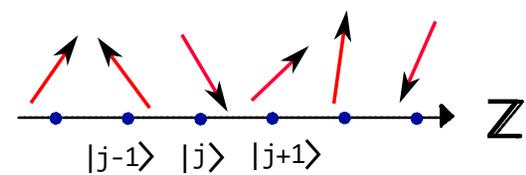
Remarks:

- The Coin Matrix $C \in U(2)$ is a parameter of the QW.

$$\text{Take } C = \begin{pmatrix} r & t \\ -t & r \end{pmatrix}, \text{ s.t. } 0 \leq r, t \leq 1 \text{ and } r^2 + t^2 = 1.$$

- $U(C)|j \otimes +1\rangle = r|(j+1) \otimes +1\rangle - t|(j-1) \otimes -1\rangle$
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- QW \rightsquigarrow RW:

Starting from $|j \otimes +1\rangle \rightsquigarrow$

$$\begin{aligned} \text{Prob } (j \rightarrow j+1) &= r^2 \\ \text{Prob } (j \rightarrow j-1) &= t^2 \end{aligned}$$

Starting from $|j \otimes -1\rangle \rightsquigarrow$

$$\begin{aligned} \text{Prob } (j \rightarrow j+1) &= t^2 \\ \text{Prob } (j \rightarrow j-1) &= r^2 \end{aligned}$$

- "Symmetric Walk": $r = t$, i.e. Hadamard Matrix $C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$U(C) = S(\mathbb{I} \otimes C)$$

Discrete Time Dynamics

Given $U(C)$ and $\Psi_0 \Rightarrow$ QW in $l^2(\mathbb{Z}) \otimes \mathbb{C}^2$

$$\Psi_0 \rightsquigarrow \Psi_1 = U(C)^1 \Psi_0 \rightsquigarrow \Psi_2 = U(C)^2 \Psi_0 \cdots \rightsquigarrow \Psi_n = U(C)^n \Psi_0 \cdots$$

QW Dynamics "on \mathbb{Z} "

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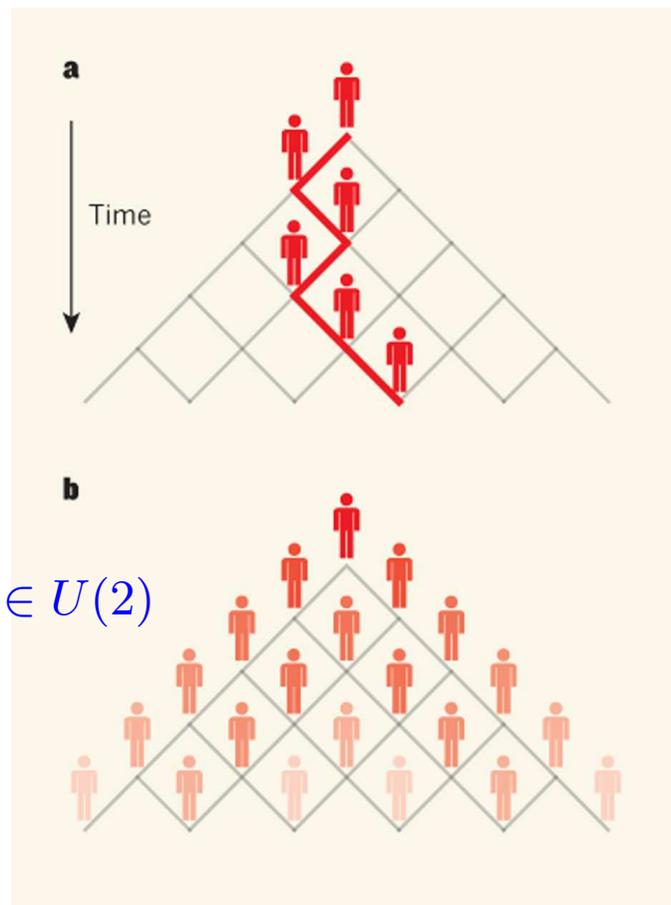
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RW vs QW

Random Walker \Rightarrow Coin



Quantum Walker \Rightarrow Coin Matrix $C \in U(2)$



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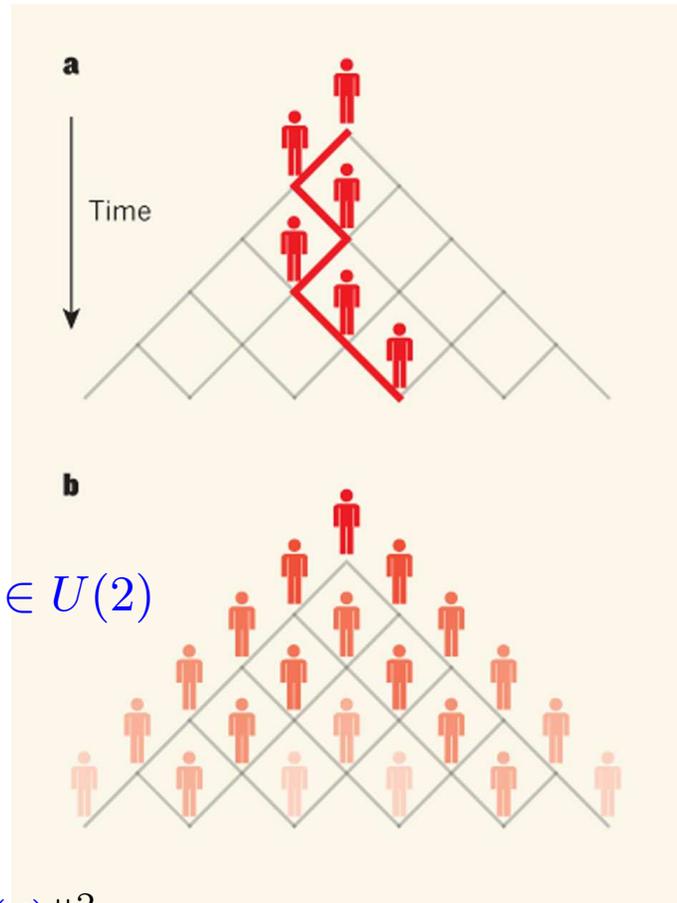
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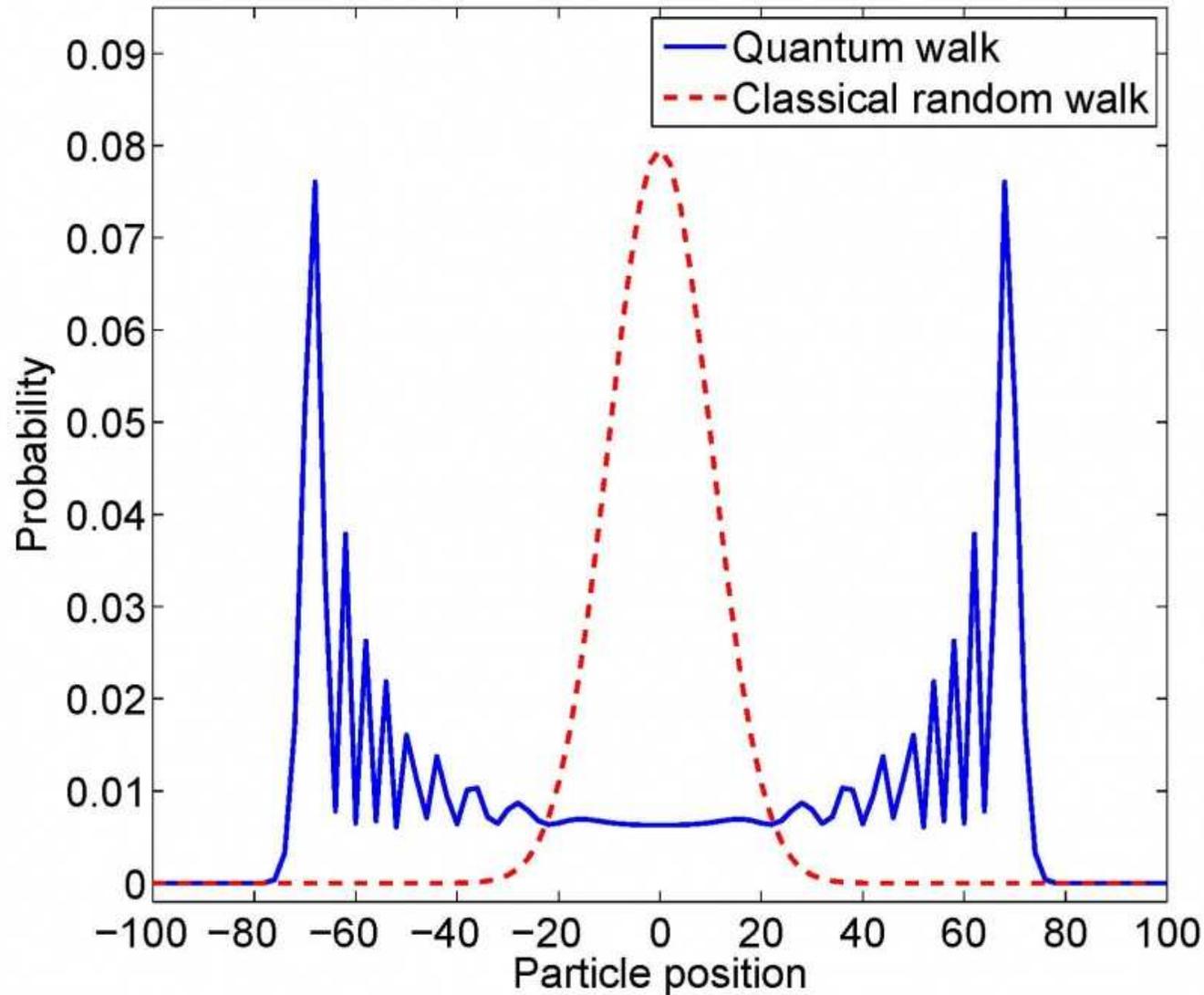
Position Distribution at time n

State at time n : $\Psi_n = U(C)^n \Psi_0 \Rightarrow$

$$\text{Prob(QW on site } j \text{ at time } n) = \|\psi_n(j)\|^2$$

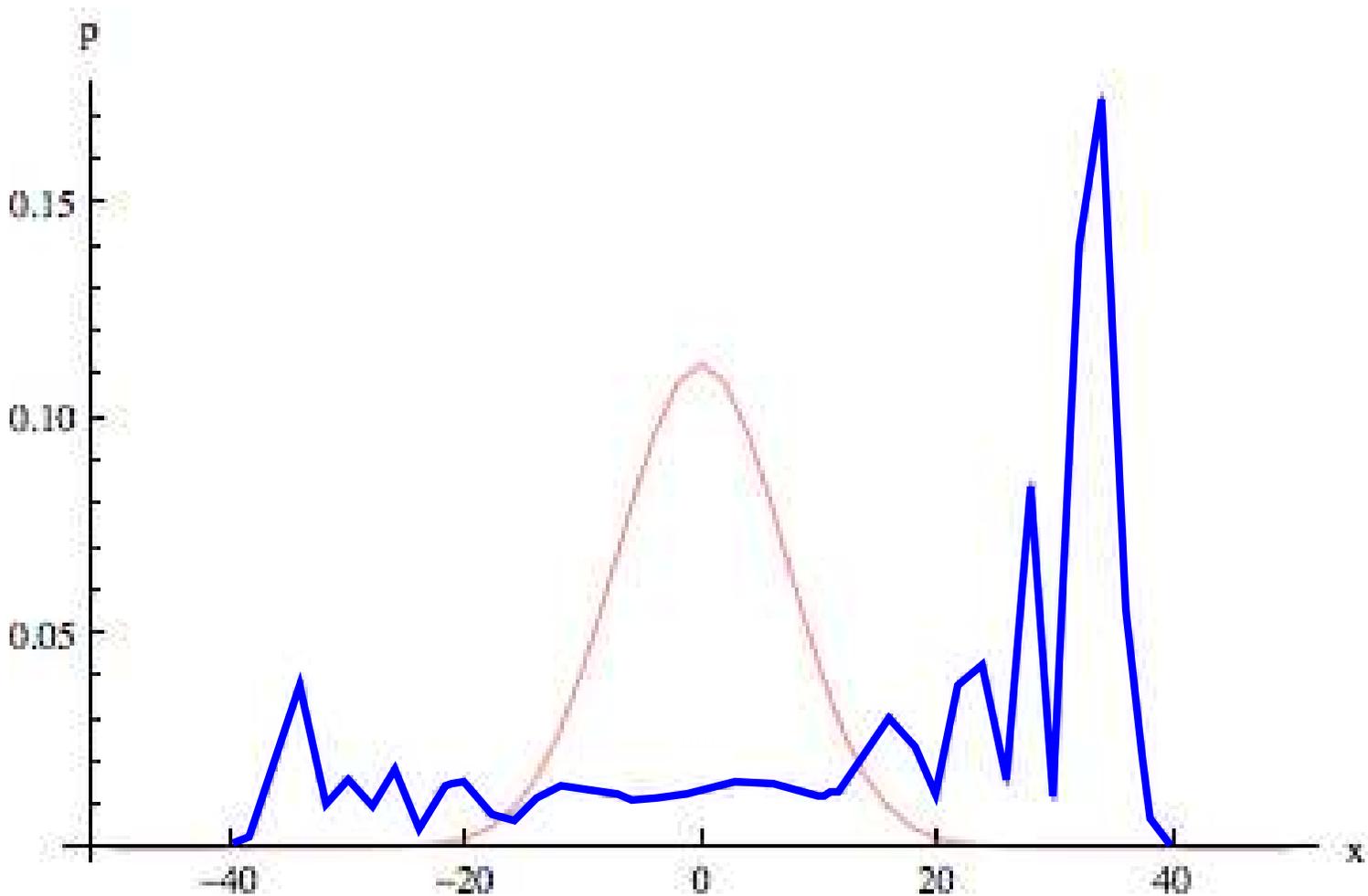
Hadamard Walk: $C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $\Psi_0 = |0\rangle \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$

Distribution of $|\Psi_n(j)|^2$ $n \simeq 75$



Hadamard Walk: $C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $\Psi_0 = |0\rangle \otimes \begin{pmatrix} a \\ b \end{pmatrix}$, $|a| \neq |b|$

Distribution of $|\Psi_n(j)|^2$ $n \simeq 40$



QM Characteristics of $\{\|\psi_n(\cdot)\|^2\}$, with $\Psi_0 = |0\rangle \otimes \varphi$

- Expectation: $\langle X_n \rangle_{\Psi_0} = \sum_{j \in \mathbb{Z}} j \|\psi_n(j)\|^2$
- Variance: $\text{Var}(X_n)_{\Psi_0} = \sum_{j \in \mathbb{Z}} (j - \langle X_n \rangle_{\Psi_0})^2 \|\psi_n(j)\|^2$, as $n \rightarrow \infty$

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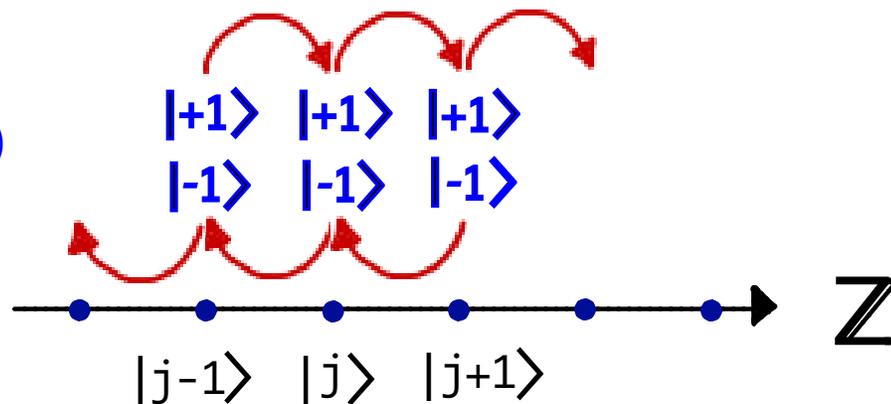
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Special Cases

- C Diagonal \Rightarrow Spin $+1/-1$ components shifted to the Left/Right

$$\langle X_n \rangle_{\Psi_0} = O(n)$$

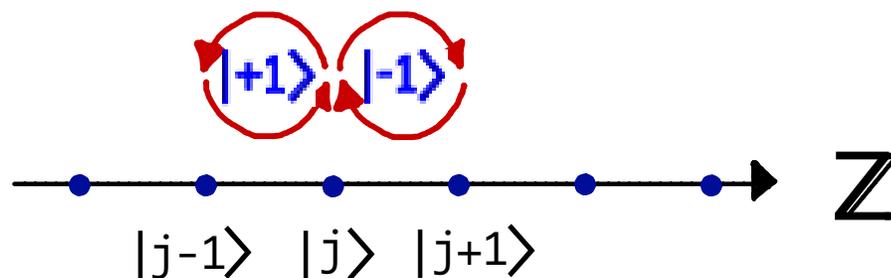
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- C Off-Diagonal \Rightarrow Quantum Walker stuck on sites $\{-1, 0, 1\}$

$$\langle X_n \rangle_{\Psi_0} = O(1)$$

$$\text{Var}(X_n)_{\Psi_0} = O(1)$$



QM Characteristics of $\{|\psi_n(\cdot)|^2\}$, with $\Psi_0 = |0\rangle \otimes \varphi$

Generic C

$\langle X_n \rangle_{\Psi_0} = O(n)$ and $\text{Var}(X_n)_{\Psi_0} = O(n^2)$, i.e. **Ballistic Behaviour**

Limiting distribution of X_n/n as $n \rightarrow \infty$.

Konno '02

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Spectral Method Transl. invar. on $\mathbb{Z} \rightsquigarrow$ Fourier transform

- $l^2(\mathbb{Z}^2) \otimes \mathbb{C}^2 \simeq L^2(\mathbb{T}_k; \mathbb{C}^2) \ni \hat{\psi}(k) = \sum_j \psi(j) e^{ijk}$
- $U(C) = S(\mathbb{I} \otimes C) \simeq \begin{pmatrix} e^{-ik} & 0 \\ 0 & e^{ik} \end{pmatrix} C \equiv C(k)$ as a $U(2)$ -valued mult. op.
 $C(k) = e^{i\omega(k)} |v_+(k)\rangle \langle v_+(k)| + e^{-i\omega(k)} |v_-(k)\rangle \langle v_-(k)|$
- $\omega(k) = \arccos(r \cos(k)) \Rightarrow \sigma(U(C)) = \sigma_{a.c.}(U(C)) \rightsquigarrow$ ballistic behavior

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- $(C^n(\cdot)\hat{\psi})(k) = f_-(k) e^{-in\omega(k)} |v_-(k)\rangle + f_+(k) |e^{+in\omega(k)} |v_+(k)\rangle, \forall n \in \mathbb{Z}$

$$\text{Var}(X_n)_{\Psi_0} = \sum_j j^2 \|\psi_n(j)\|^2 = n^2 \int_{\mathbb{T}_k} (\partial_k \omega(k))^2 \|\hat{\psi}(k)\|^2 + O(n)$$

Group velocity: $\partial_k \omega(k)$

Random Quantum Walk: Static Disorder

Site-dep. Coin Matrices $C \rightsquigarrow \{C(j)\}_{j \in \mathbb{Z}}$

$$U|j \otimes \pm 1\rangle = C_{+,\pm}(j)|(j+1) \otimes +1\rangle + C_{-,\pm}(j)|(j-1) \otimes -1\rangle$$

Spatial inhomogeneity Coin $C^\omega(j)$ at each site j is **random**

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Assumptions:

J.-Merkli '09

- $\{C_\omega(k)\}_{k \in \mathbb{Z}}$ is an i.i.d. $U(2)$ -valued random variable
- The quantum transition **amplitudes** to the right and left are **independent**
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Consequence

Lemma: Up to trivial transformations,

$\exists r, t \in [0, 1]$ s.t. $r^2 + t^2 = 1$ and i.i.d random phases $\{\omega_k^\pm\}_{k \in \mathbb{Z}}$

$$C_\omega(k) := \begin{pmatrix} r e^{-i\omega_k^+} & t e^{-i\omega_k^+} \\ -t e^{-i\omega_k^-} & r e^{-i\omega_k^-} \end{pmatrix}, \quad k \in \mathbb{Z}.$$

Random Quantum Walk: Static Disorder

Structure

$$U(C) \mapsto U_{\omega}(C)$$

Property: Let $\{|k\rangle \otimes |\tau\rangle\}_{k \in \mathbb{Z}}^{\tau \in \{\pm\}}$ be ONB of $l^2(\mathbb{Z}) \otimes \mathbb{C}^2$
Set $\mathbb{D}(\omega) = \text{diag}(\exp(i\omega_k^\tau))$, then **random** time one dynamics

$$U_{\omega}(C) = \mathbb{D}(\omega)U(C)$$

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Thm "Anderson Localization"

J.-Merkli '09, Ahlbrecht et al '11

If ω_k^\pm have an L^∞ density,

For any $r \in [0, 1)$, any $p \in \mathbb{N}$, any Ψ_0 of cpct. supp. :

$$\sup_{n \in \mathbb{Z}} \langle |X_n^\omega|^p \rangle_{\Psi_0} < C_\omega, \text{ a.s.}$$

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First RQW: '09 Konno:

$$C^\omega(k) = \begin{pmatrix} r e^{i\omega k} & t \\ -t & r e^{-i\omega k} \end{pmatrix} \Rightarrow \text{ballistic behaviour!}$$

Random Quantum Walk: Static Disorder

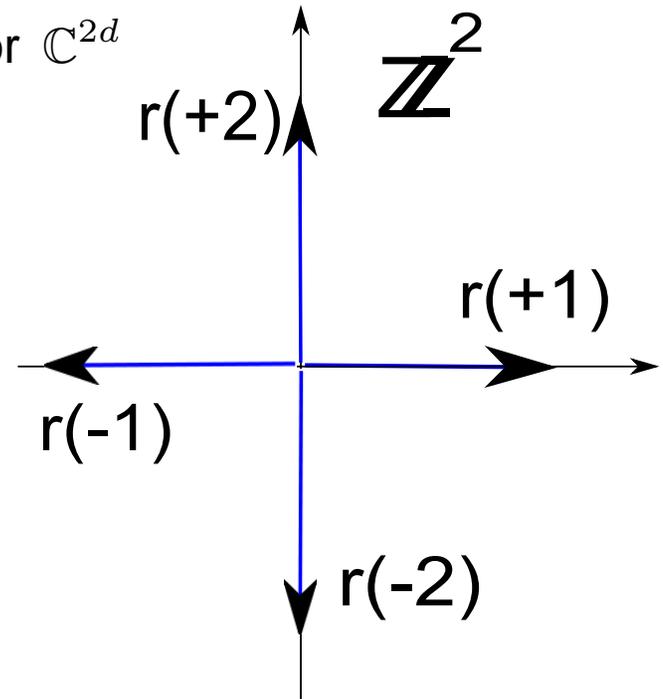
Generalization: RQW on $\mathbb{Z}^d \rightsquigarrow \mathcal{H} = l^2(\mathbb{Z}^d) \otimes \mathbb{C}^{2d}$

J. '12

ONB: $\{|\tau\rangle\}_{\tau \in I_{\pm}}$, $I_{\pm} \equiv \{\pm 1, \pm 2, \dots, \pm d\}$ for \mathbb{C}^{2d}

Jump Function $r : I_{\pm} \rightarrow \mathbb{Z}^d$ s.t. $\tau \mapsto r(\tau)$

$$S = \sum_{\tau \in I_{\pm}} \sum_{k \in \mathbb{Z}^d} |k + r(\tau)\rangle \langle k| \otimes P_{\tau}$$



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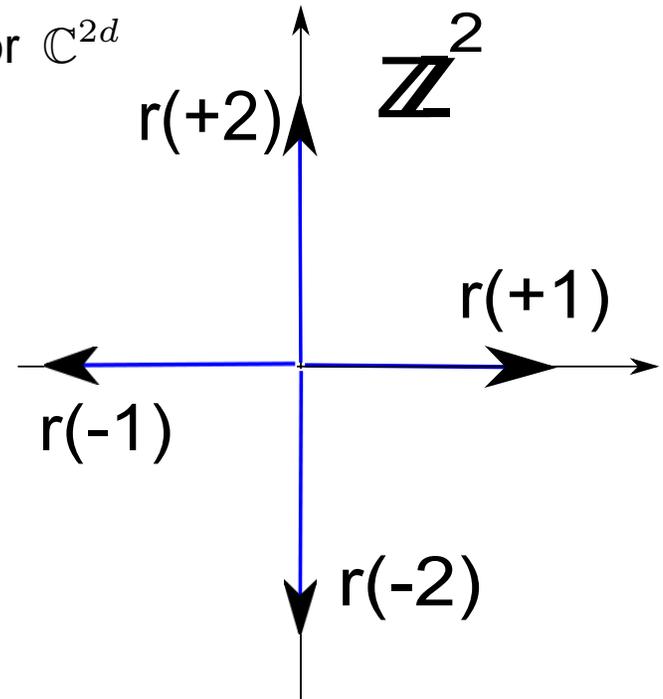
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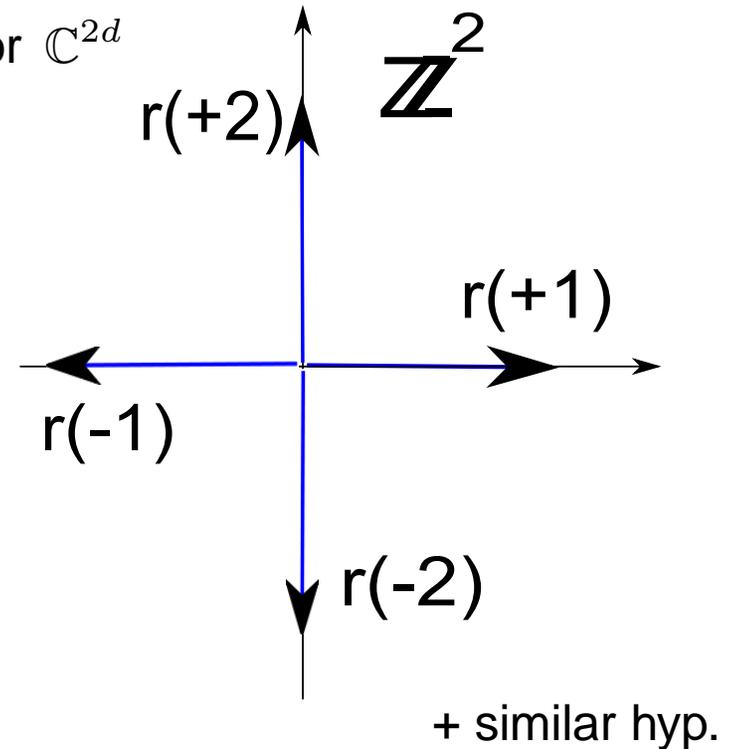
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"Large Disorder Regime" Anderson Localization

If C close to certain Permutation matrices



$$\boxed{\sup_{n \in \mathbb{Z}} \langle |X_n^{\omega}|^p \rangle_{\psi_0} < C_{\omega}, \text{ a.s.}}$$

Rem:

Permut. matrices \iff Off-Diag C when $d = 1$ \iff Stuck Quantum Walker

Random Quantum Walk: Temporal Disorder

Temporal disorder

New coin C^ω at each time step:

Kosk et al '06

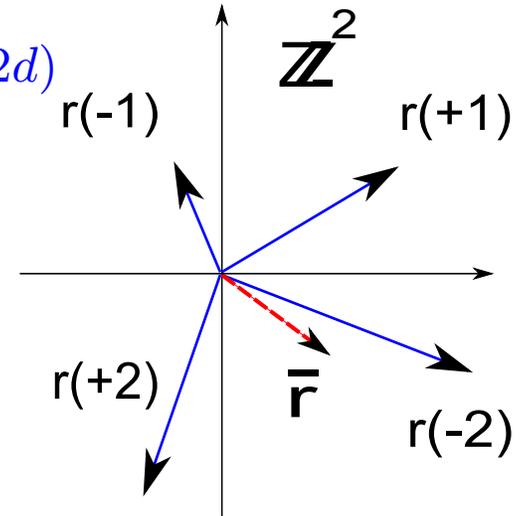
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Random Coin op's

$\{C^\omega(k)\}_{k \in \mathbb{N}}$, i.i.d. with common distribution μ on $U(2d)$

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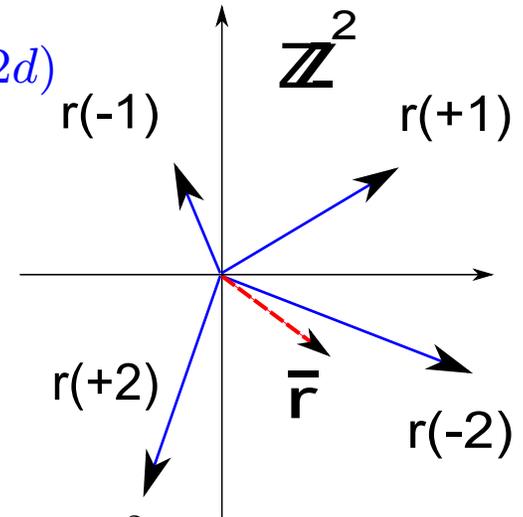
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Averaging over randomness of QM expectations:

$$\langle \overline{X_n} \rangle_{\Psi_0} = \mathbb{E}_\omega(\langle X_n^\omega \rangle_{\Psi_0}) \Leftrightarrow w_n(\cdot) = \mathbb{E}_\omega(\|\psi_n^\omega(\cdot)\|^2)$$



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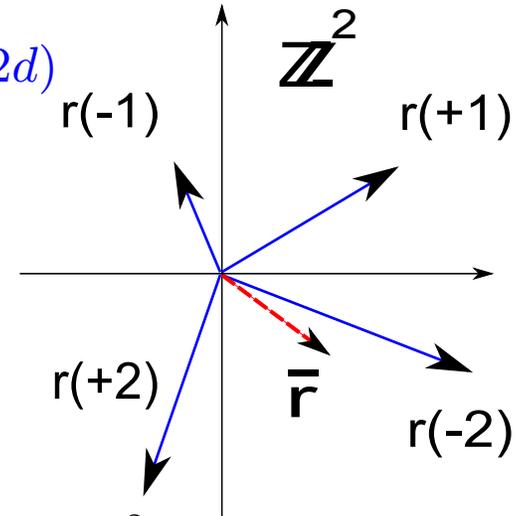
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Thm "Diffusive Behaviour"

Ahlbrecht et al '11, J.'11, Hamza-J.'12

$$\lim_{n \rightarrow \infty} \frac{\langle \overline{X_n} \rangle_{\Psi_0}}{n} = \mu$$

Drift

$$\lim_{n \rightarrow \infty} \frac{\langle (X_n - n\mu)^2 \rangle_{\Psi_0}}{n} = D$$

Diffusion Const.

Summary

Coined Random Quantum Walks on \mathbb{Z}^d

Generic situation

- C uniform \Rightarrow ballistic behaviour
- C^ω random in space \Rightarrow a.s. localization
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Remarks

- C^ω random in time and space
 \Rightarrow average diffusive behaviour Hamza-J. 12, Ahlbrecht et al '13
- (R)QW on \mathbb{Z} and CMV matrices: special cases of discrete dyn. systems
generated by 5-diag. unitary operators on $l^2(\mathbb{Z})$ Bourget-Howland-J. '03
- Warning \exists a notion of localization for QW \iff point spectrum in $U(C)$.

More on Random Quantum Walk II

Random evol.

$$U_{\omega}(n, 0) = S(\mathbb{I} \otimes C_{\omega}(n)) \cdots S(\mathbb{I} \otimes C_{\omega}(2)) S(\mathbb{I} \otimes C_{\omega}(1))$$

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$$\begin{aligned}
 U_\omega(n, 0) &= S(\mathbb{I} \otimes C_\omega(n)) \cdots S(\mathbb{I} \otimes C_\omega(2)) S(\mathbb{I} \otimes C_\omega(1)) \\
 &= \sum_{\substack{\tau_n, \dots, \tau_1 \\ \tau_j \in I_\pm}} \sum_{x \in \mathbb{Z}^d} |x + \sum_{j=1}^n r(\tau_j)\rangle \langle x| \otimes P_{\tau_n} C_\omega(n) \cdots P_{\tau_1} C_\omega(1) \\
 &\equiv \sum_{x \in \mathbb{Z}^d} \sum_{|k| \leq \rho n} |x + k\rangle \langle x| \otimes J_k^\omega(n), \quad \rho = \max_{\tau \in I_\pm} |r(\tau)|
 \end{aligned}$$

where

$$J_k^\omega(n) = \sum_{\substack{\tau_n, \dots, \tau_1 \\ \sum_{j=1}^n r(\tau_j) = k}} P_{\tau_n} C_\omega(n) \cdots P_{\tau_1} C_\omega(1) \in M_{2d}(\mathbb{C})$$

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$$| \tau_n \rangle \langle \tau_n | C_\omega(n) | \tau_{n-1} \rangle \cdots | \tau_1 \rangle \langle \tau_1 | C_\omega(1)$$

Note

$$\Psi_n^\omega = U_\omega(n, 0)(|0\rangle \otimes \varphi_0) = \sum_{|k| \leq \rho n} |k\rangle \otimes J_k^\omega(n) \varphi_0 \equiv \sum_{|k| \leq \rho n} |k\rangle \otimes \psi_n^\omega(k)$$

QM Probability distribution

Lattice observables

If $f : \mathbb{Z}^d \rightarrow \mathbb{C}$, let $F := \mathbb{I} \otimes f$ on $D(F) \subset \mathcal{H}$ s.t.

$$F(\varphi_0 \otimes |k\rangle) = f(k)(\varphi_0 \otimes |k\rangle), \quad \forall \varphi_0 \in \mathbb{C}^{2d}.$$

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QM Expectation Value Let F , $\Psi_0 = \varphi_0 \otimes |0\rangle$, $\Psi_n^\omega = U_\omega(n, 0)\Psi_0$

$$\langle F \rangle_{\Psi_0}^\omega(n) = \sum_{k \in \mathbb{Z}^d} f(k) \|\psi_n^\omega(k)\|^2,$$

with $\|\psi_n^\omega(k)\|^2 = \|J_k^\omega(n)\varphi_0\|_{\mathbb{C}^{2d}}^2$

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$$\langle F \rangle_{\Psi_0}^\omega(n) = \mathbb{E}_{W_k^\omega(n)}(f(X_n^\omega)).$$

Averaged QM Probability distribution

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$$\text{with } \mathbb{E}_\omega \|\psi_n^\omega(k)\|^2 = \mathbb{E}_\omega \|J_k^\omega(n)\varphi_0\|_{\mathbb{C}^{2d}}^2$$

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Determin. Distribution Let X_n be a RV on \mathbb{Z}^d s.t.

$$\text{Prob}(X_n = k) = w_k(n) = \mathbb{E}_\omega \|J_k^\omega(n)\varphi_0\|_{\mathbb{C}^{2d}}^2,$$

$$\overline{\langle F \rangle}_{\Psi_0}(n) = \mathbb{E}_{w_k(n)}(f(X_n)).$$

Goal Understand $\{w_k(n)\}_{k \in \mathbb{Z}^d}$ as $n \rightarrow \infty$

Ballistic vs. Diffusive Scaling

Expected

- r -dependent Drift: $X_n \simeq vn$
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$\alpha = 1 \Leftrightarrow$ ballistic scaling $\alpha = 1/2 \Leftrightarrow$ diffusive scaling

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Rem: Charact. Funct. $\Phi_n(y)$ is analytic in \mathbb{C}^d .

Black Box Reformulation

Steps

- Expansion of $w_k(n) = \mathbb{E}_\omega \|J_k^\omega(n)\varphi_0\|_{\mathbb{C}^{2d}}^2 \rightsquigarrow$ sums over **weighted paths**
- Extended space $\mathbb{C}^d \mapsto \mathbb{C}^d \otimes \mathbb{C}^d$
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Set for any $(y, y') \in \mathbb{T}^d \times \mathbb{T}^d$

- $D(y, y') := d(y) \otimes d(y')$, where $d(y) = \sum_{\tau \in I_\pm} e^{iyr(\tau)} |\tau\rangle\langle\tau|$,
- $\mathcal{E} := \mathbb{E}_\omega (C_\omega \otimes \overline{C_\omega})$, $\mathcal{M}(y, y') := D(y, y')\mathcal{E}$
- $\Psi_1 = \sum_{\tau \in I_\pm} |\tau \otimes \tau\rangle$, $\Phi_0 = \varphi_0 \otimes \overline{\varphi_0}$

Char. Funct.

For $y \in \mathbb{T}^d$,

$$\Phi_n(y) = \int_{\mathbb{T}^d} \langle \Psi_1 | \mathcal{M}^n(y - v, v) \Phi_0 \rangle d\tilde{v}$$

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Properties For all $v \in \mathbb{T}^d$

- $\Psi_1 = \sum_{\tau \in I_{\pm}} |\tau \otimes \tau\rangle$ invar. under $\mathcal{M}(-v, v)$ and its adjoint
- $\text{Spr } \mathcal{M}(-v, v) = \|\mathcal{M}(-v, v)\| = 1$

Spectral Analysis

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Consequently $\exists 0 < \delta < 1$, a complex ngbhd. \mathcal{N} of $\{0\} \times \mathbb{T}^d$ s.t. $\forall (y, v) \in \mathcal{N}$

$$\sigma(\mathcal{M}(y - v, v)) \cap D(1, \delta) = \{\lambda_1(y, v)\}$$

$$\sigma(\mathcal{M}(y - v, v)) \setminus \{\lambda_1(y, v)\} \subset D(0, 1 - \delta)$$

$$\mathcal{M}(y - v, v) = \lambda_1(y, v)P(y, v) + \mathcal{R}(y, v), \quad \text{Spr}(\mathcal{R}(y, v)) < 1 - \delta$$

where $\lambda_1(y, v)$, $P(y, v)$, $\mathcal{R}(y, v)$ are **analytic** in \mathcal{N} and

$$\lambda_1(0, v) \equiv 1 \text{ and } P(0, v) \equiv |\Psi_1\rangle\langle\Psi_1|/(2d).$$

Perturbation Theory

$$\Phi_n(y) = \int_{\mathbb{T}^d} \langle \Psi_1 | \mathcal{M}^n(y - v, v) \Phi_0 \rangle d\tilde{v}$$

Diffusive/Ballistic Scaling $\Rightarrow y \mapsto y/n^\alpha \ll 1$

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Analytic Perturbation Theory

For $(0, v) \in \mathcal{N}$,

$$\lambda_1(y, v) \equiv 1 + y \frac{i}{2d} \sum_{\tau \in I_\pm} r(\tau) - \frac{1}{2} \langle y | \mathbb{D}(v) y \rangle + O_v(\|y\|^3)$$

The map $v \mapsto \mathbb{D}(v) \in M_d(\mathbb{C})$ is **real analytic** in \mathbb{T}^d and $\mathbb{D}(v)_{j,k} = \frac{\partial^2}{\partial y_j \partial y_k} \lambda(0, v)$

For $v \in \mathbb{T}^d$, $\mathbb{D}(v) \geq 0$ and $O_v(\|y\|^3)$ is unif. in v in a complex nbhd. of \mathbb{T}^d .

Results in Average

Assumption S $\Rightarrow \exists$ a real analytic map

$$\mathbb{T}^d \ni v \mapsto \mathbb{D}(v) \in M_d(\mathbb{R}), \text{ s.t. } \mathbb{D}(v) \geq 0.$$

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$$\lim_{n \rightarrow \infty} \Phi_n(y/n) = e^{iy\bar{r}}, \quad \bar{r} = \frac{1}{2d} \sum_{\tau \in I_{\pm}} r(\tau)$$

$$\lim_{n \rightarrow \infty} e^{-in \frac{\bar{r}y}{\sqrt{n}}} \Phi_n(y/\sqrt{n}) = \int_{\mathbb{T}^d} e^{-\frac{1}{2} \langle y | \mathbb{D}(v) y \rangle} d\tilde{v}.$$

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Consequently

$$\lim_{n \rightarrow \infty} \frac{\langle X_i \rangle_{\psi_0}(n)}{n} = \bar{r}_i$$

Drift

$$\lim_{n \rightarrow \infty} \frac{\langle (X - n\bar{r})_i (X - n\bar{r})_j \rangle_{\psi_0}(n)}{n} = \int_{\mathbb{T}^d} \mathbb{D}_{ij}(v) d\tilde{v}$$

Diffusion Matrix.

+ Moderate/Large Deviations Results

Variants / Perspectives

Variations

- Open Quantum Walks
- Interacting Quantum Walkers
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Perspectives

- Delocalization regime for the Chalker-Coddington model ?
- Statistics of eigenvalues for RQW on \mathbb{Z}^d ?
- (De-)localization properties of Random Open Quantum Walks ?
- ...