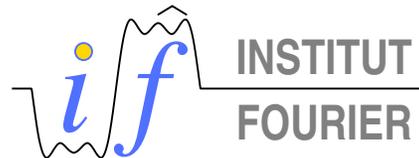


Spectral Transition for Random Quantum Walks on Trees*

Alain JOYE



* *Joint work with Eman HAMZA, Cairo University*

Guideline: Quantum Walk on $\mathbb{Z} = \mathcal{T}_2$

Quantum particle with spin 1/2 on 1-dim lattice, i.e. $\mathcal{K}_2 = l^2(\mathbb{Z}) \otimes \mathbb{C}^2$

Spin evol.: C unitary op. on \mathbb{C}^2 , “coin” space

Spin dependent shift:

Let S_{\pm} shift to the right/left on $l^2(\mathbb{Z})$, $|\pm\rangle\langle\pm|$ proj. on $|\pm\rangle \in \mathbb{C}^2$

$S := S_+ \otimes |+\rangle\langle+| + S_- \otimes |-\rangle\langle-|$ on $l^2(\mathbb{Z}) \otimes \mathbb{C}^2$

One step dynamics: $U := S(\mathbb{I} \otimes C)$ s.t.

$$U|x \otimes \pm\rangle = \langle-|C \pm\rangle|(x-1) \otimes -\rangle + \langle+|C \pm\rangle|(x+1) \otimes +\rangle$$

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Random Quantum Walk:

$C \rightsquigarrow \{C_{\omega}(x)\}_{x \in \mathbb{Z}}$, i.i.d. set of $C_{\omega}(x) \in U(2)$

$$U \rightsquigarrow U_{\omega}|x \otimes \pm\rangle = \langle-|C_{\omega}(x) \pm\rangle|(x-1) \otimes -\rangle + \langle+|C_{\omega}(x) \pm\rangle|(x+1) \otimes +\rangle$$

Homogeneous tree \mathcal{T}_3 with coord. number $q = 3$

$A_3 = \{a, b, c\}$ generators of a group with unit e s.t. $a^2 = b^2 = c^2 = e$.

Root: Origin e

Edges: Labels $\{a, b, c\}$ in trig. order

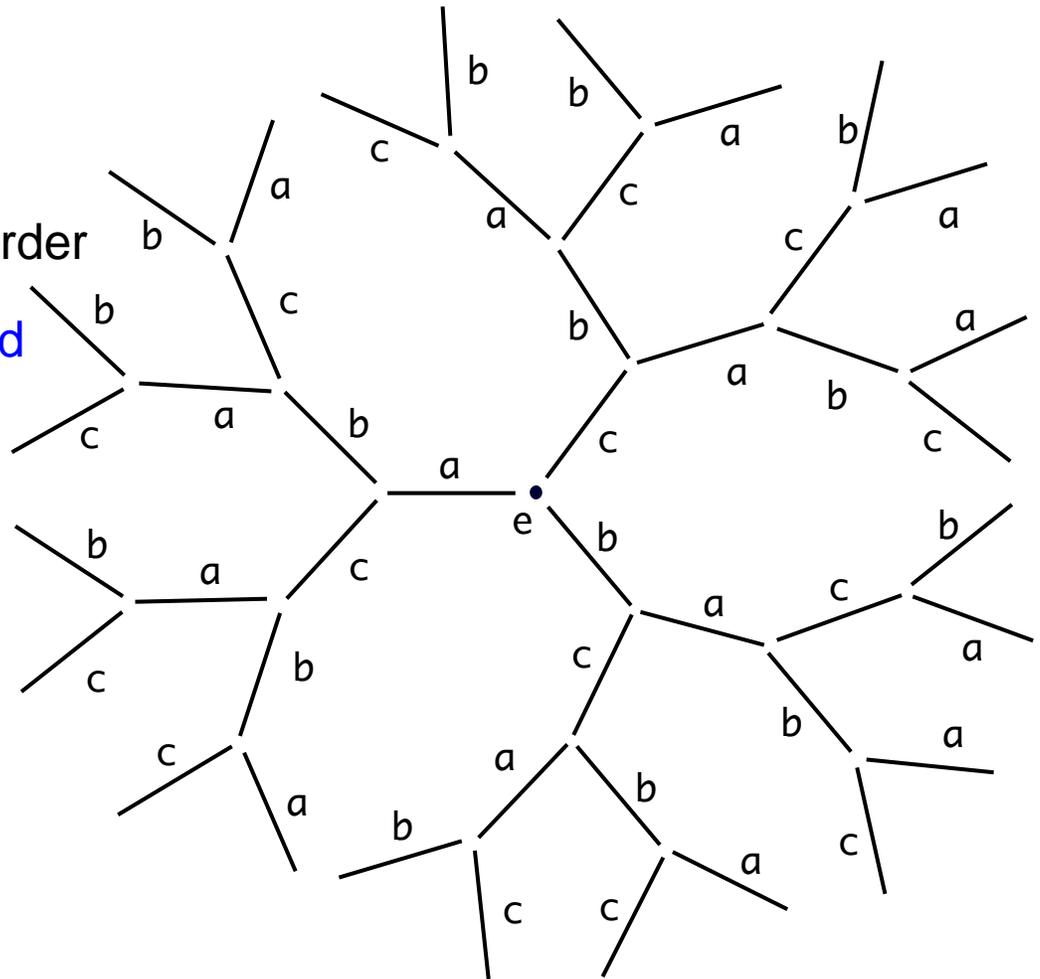
Any $x \in \mathcal{T}_3$: Finite reduced word

$$x = x_1 x_2 \cdots x_n, x_j \in A_3$$

Length: $|x| = n$

Distance: For any $x, y \in \mathcal{T}_3$,

$$d(x, y) = |x^{-1}y|$$



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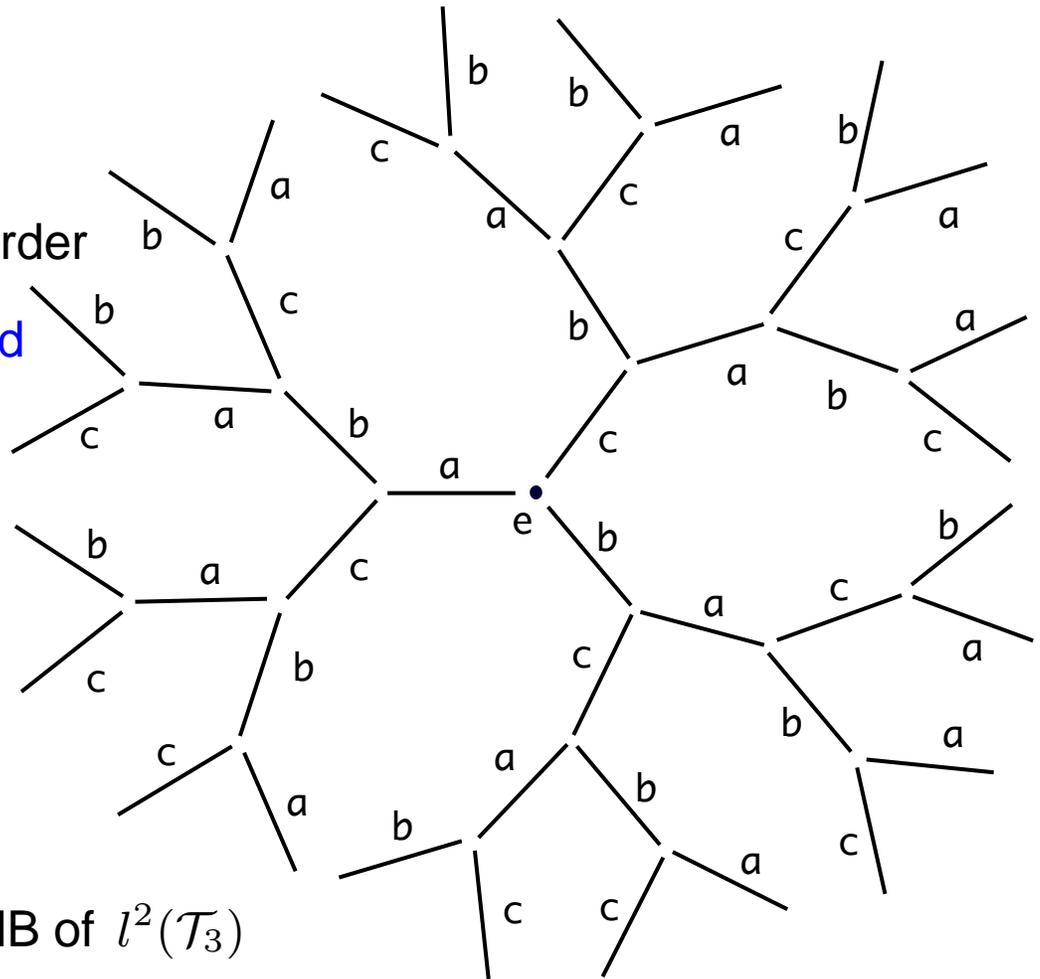
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$l^2(\mathcal{T}_3)$:

$x \mapsto |x\rangle$, s.t. $\{|x\rangle\}_{x \in \mathcal{T}_3}$ ONB of $l^2(\mathcal{T}_3)$

$$\psi = \sum_{x \in \mathcal{T}_3} \psi(x) |x\rangle, \quad \sum_{x \in \mathcal{T}_3} |\psi(x)|^2 < \infty.$$



Shifts on \mathcal{T}_3

Even / Odd: Sites x_e / x_o s.t. $|x_e|$ even / $|x_o|$ odd

Let $S_{ab} : l^2(\mathcal{T}_3) \rightarrow l^2(\mathcal{T}_3)$

$$S_{ab}|x_e\rangle = |x_e a\rangle$$

$$S_{ab}|x_o\rangle = |x_o b\rangle$$

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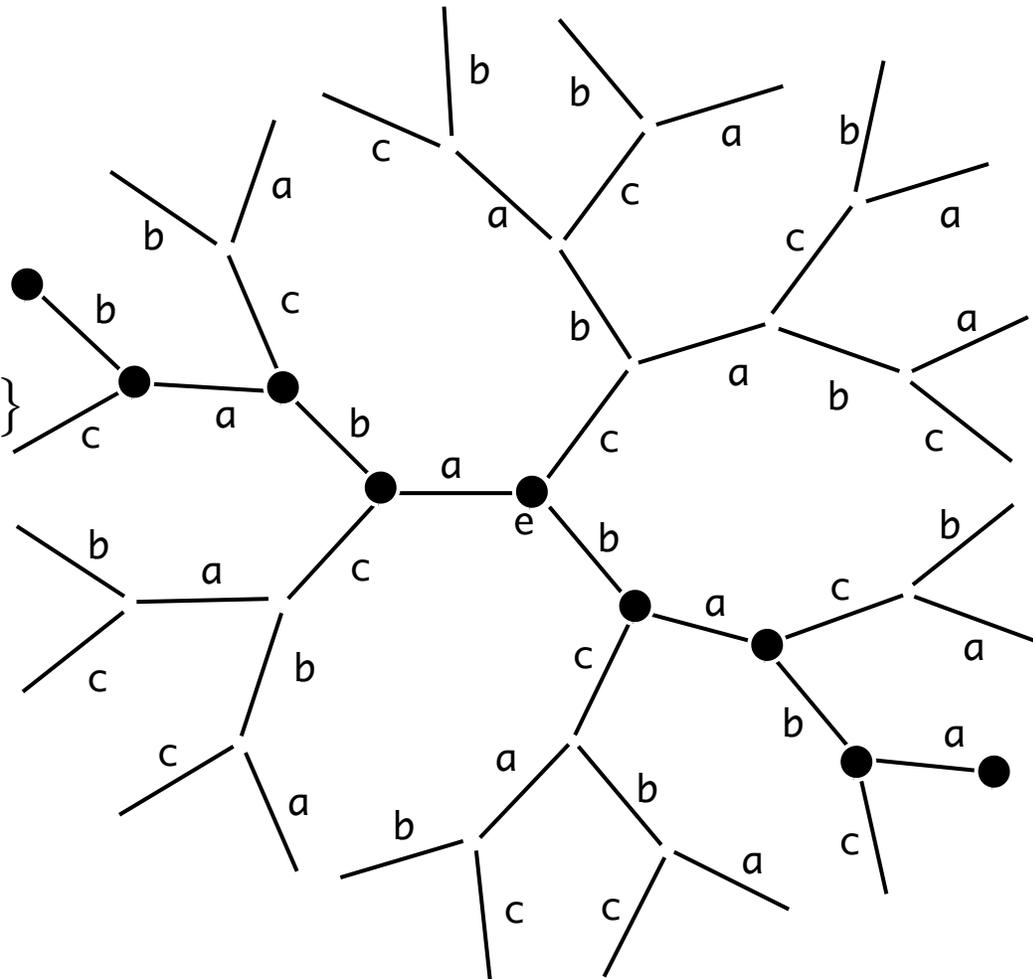
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Cyclic subspaces:

$$\mathcal{H}_x^{ab} = \text{span} \{ S_{ab}^n |x\rangle, n \in \mathbb{Z} \}$$

$$S_{ab}|_{\mathcal{H}_x^{ab}} \simeq \text{shift on } l^2(\mathbb{Z}).$$



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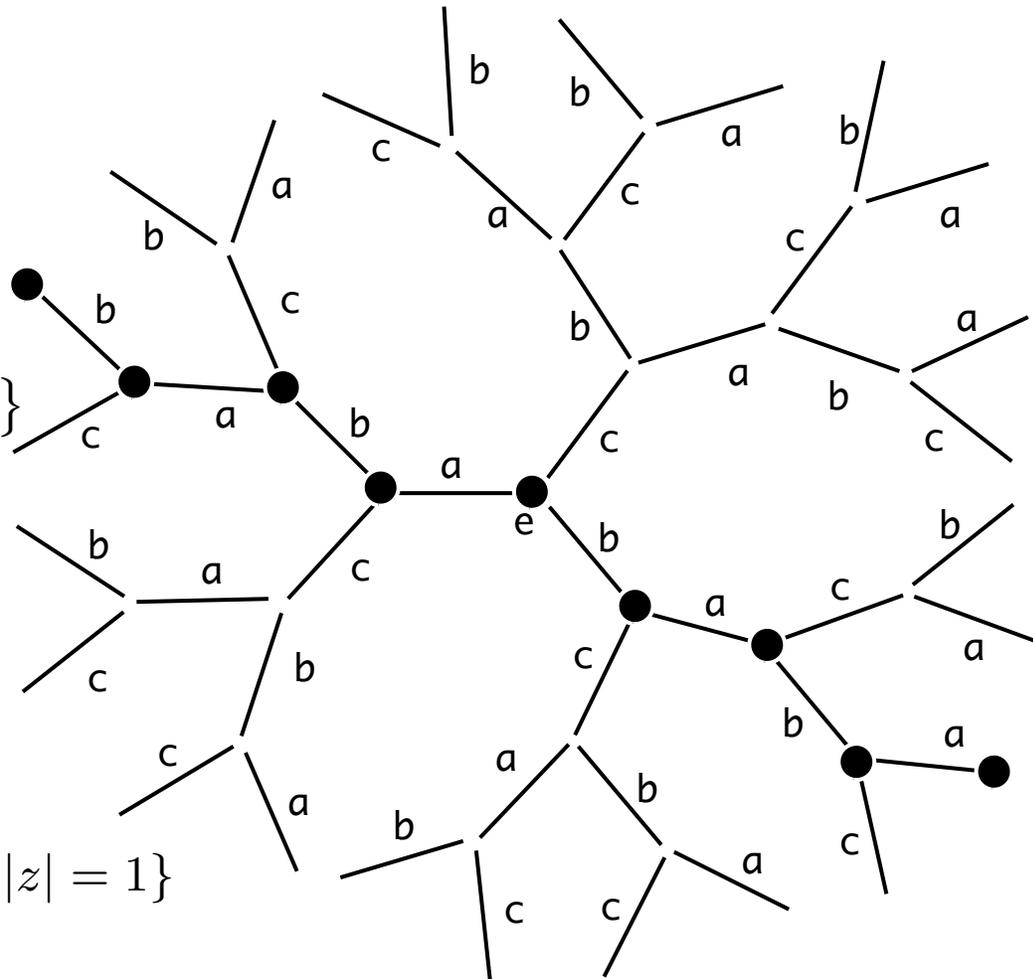
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Spectrum:

$$\sigma(S_{ab}) = \sigma_{ac}(S_{ab}) = \{z \text{ s.t. } |z| = 1\}$$

Similarly for S_{bc} , S_{ca}



Quantum Walk on \mathcal{T}_3

- Unitary evolution:

Particle with spin 1 on \mathcal{T}_3 jumping on nearest neighbors

Hilbert space: $\mathcal{K}_3 = l^2(\mathcal{T}_3) \otimes \mathbb{C}^3$

ONB of \mathbb{C}^3 : $\{|a\rangle, |b\rangle, |c\rangle\}$,

ONB of \mathcal{K}_3 : $\{|x\rangle \otimes |\tau\rangle\}_{x \in \mathcal{T}_3, \tau \in A_3}$

- Spin dep. shift on \mathcal{K}_3 :

$$S = S_{bc} \otimes |a\rangle\langle a| + S_{ca} \otimes |b\rangle\langle b| + S_{ab} \otimes |c\rangle\langle c|$$

- Spin update: For $C \in U(3)$ a unitary op. on \mathbb{C}^3

$$\mathbb{I} \otimes C : \mathcal{K}_3 \rightarrow \mathcal{K}_3$$

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- Time one dynamics of the QW:

$$\boxed{U(C) := S(\mathbb{I} \otimes C)}$$

s.t.

$$U(C)x_e \otimes a = C_{aa}x_e b \otimes a + C_{ba}x_e c \otimes b + C_{ca}x_e a \otimes c$$

$$U(C)x_o \otimes a = C_{aa}x_o c \otimes a + C_{ba}x_o a \otimes b + C_{ca}x_o b \otimes c \text{ etc...}$$

Random Environment \Rightarrow RQW

Spatial disorder Different coin op. C_ω at each site:

$$C \mapsto \{C_\omega(x)\}_{x \in \mathcal{T}_3} \quad \text{with} \quad \begin{aligned} C_\omega(x_e)_{\tau,\sigma} &= \exp(i\omega_{x_e\tau}^\tau) C_{\tau,\sigma}, \\ C_\omega(x_o)_{\tau,\sigma} &= \exp(i\omega_{x_o\sigma}^\tau) C_{\tau,\sigma}, \quad \tau, \sigma \in \{a, b, c\}. \end{aligned}$$

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$$U(C) \mapsto U_\omega(C)$$

Property: Set $\mathbb{D}(\omega) = \text{diag}(\exp(i\omega_x^\tau))$, then random time one dynamics

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Assumptions:

- $\{\omega_x^\tau\}_{x \in \mathcal{T}_3}^{\tau \in A_3}$ are i.i.d. \mathbb{T} -valued random variables
 $\omega_x^\tau(\omega) \simeq d\mu(\theta) = l(\theta)d\theta$ with $l \in L^\infty$

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Remarks:

- The transition prob. $|\langle x \otimes \tau | U_\omega(C) y \otimes \sigma \rangle|^2$ are deterministic.
- Spectral transition expected between "large" and "small" disorder regimes,
Abou-Chacra, Anderson, Thouless '73, Kunz Souillard '83, Klein '94

Landmarks in $U(3)$

Permutation mat. For $\pi \in \mathfrak{S}_3$ on $A_3 = \{a, b, c\}$ set $C_\pi = \sum_{\tau \in A_3} |\pi(\tau)\rangle\langle\tau|$

Consider

$$U_\omega(C_\pi) = \mathbb{D}(\omega)U(C_\pi)$$

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• $C_{(a)(b)(c)} = \mathbb{I} \quad \Rightarrow U_\omega(\mathbb{I}) = \mathbb{D}(\omega)S \simeq S$ a.c. \leftrightarrow deloc.

• $C_{(abc)} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \Rightarrow U_\omega(C_{(abc)})$ p.p. \leftrightarrow loc. (as is $U_\omega(C_{(acb)})$)

$\mathcal{H}_{x_0} = \text{span}\{x_0 \otimes a, x_0 a \otimes b, x_0 \otimes c, x_0 c \otimes a, x_0 \otimes b, x_0 b \otimes c\}$

invariant under $U_\omega(C_{(abc)})$, $\forall x_0 \in \mathcal{T}_3$

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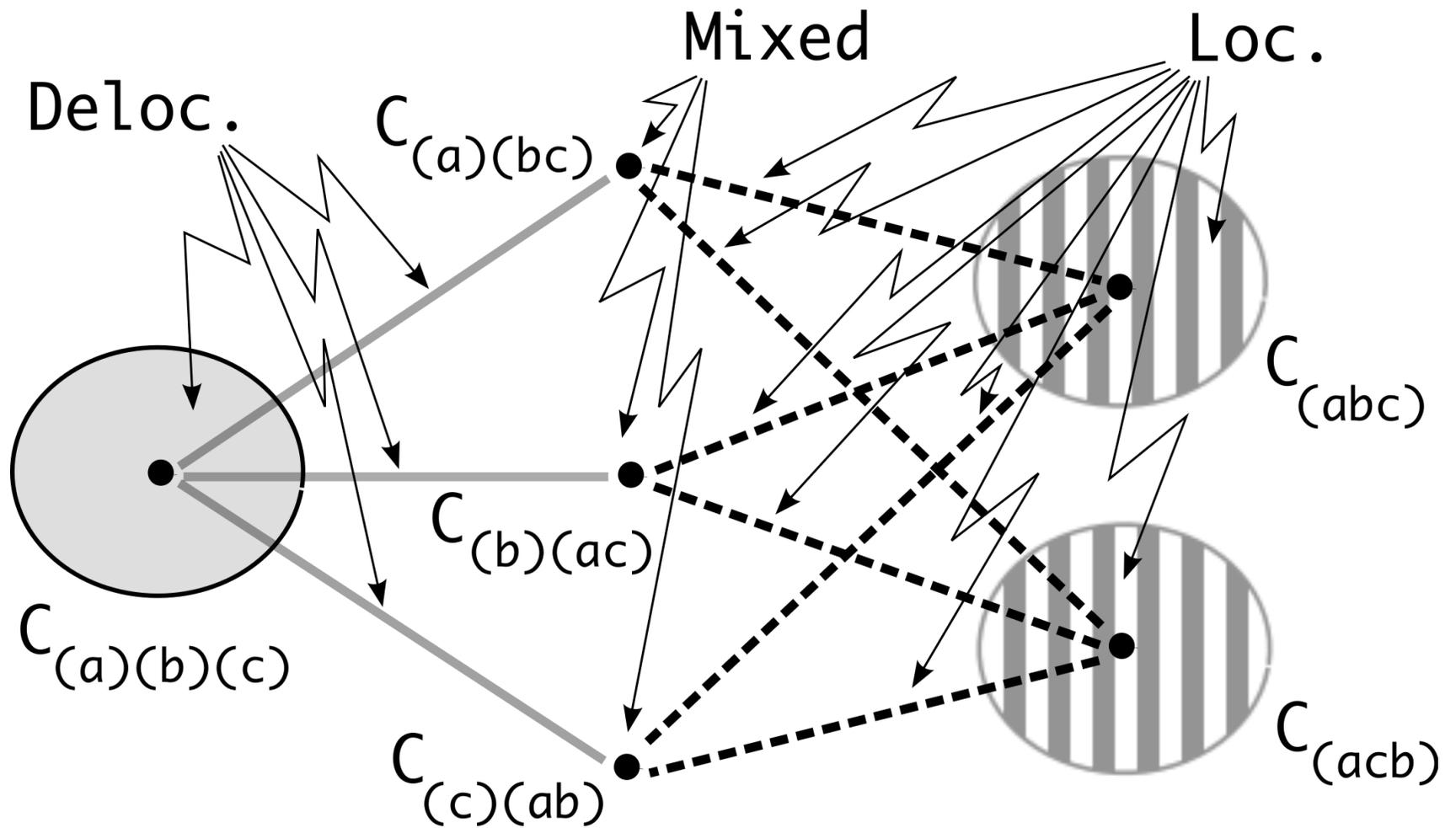
- $C_{(c)(ab)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow U_\omega(C_{(c)(ab)}) \simeq$ p.p. \oplus a.c. \leftrightarrow mixed
(as are $U_\omega(C_{(a)(bc)})$, $U_\omega(C_{(b)(ac)})$)

$$\mathcal{H}_{x_0} = \text{span}\{\dots, x_0 ba \otimes a, x_0 b \otimes b, x_0 \otimes a, x_0 a \otimes b, x_0 ab \otimes a, \dots\}$$

$$\mathcal{H}_{x_e} = \text{span}\{x_e \otimes a, x_e c \otimes b\}$$
 invar. under $U_\omega(C_{(ab)(c)})$, $\forall x_e, x_0 \in \mathcal{T}_3$

Spectral Phase Diagram

Theorem:



Interpolating Matrices I

Localizing Matrices: For $0 \leq r \leq 1$ and $t = \sqrt{1 - r^2}$,

$$C_1^l(r) = \begin{pmatrix} 0 & r & t \\ 1 & 0 & 0 \\ 0 & t & -r \end{pmatrix} \text{ s.t. } C_1^l(1) \simeq C_{(c)(ab)} \text{ and } C_1^l(0) = C_{(abc)}$$

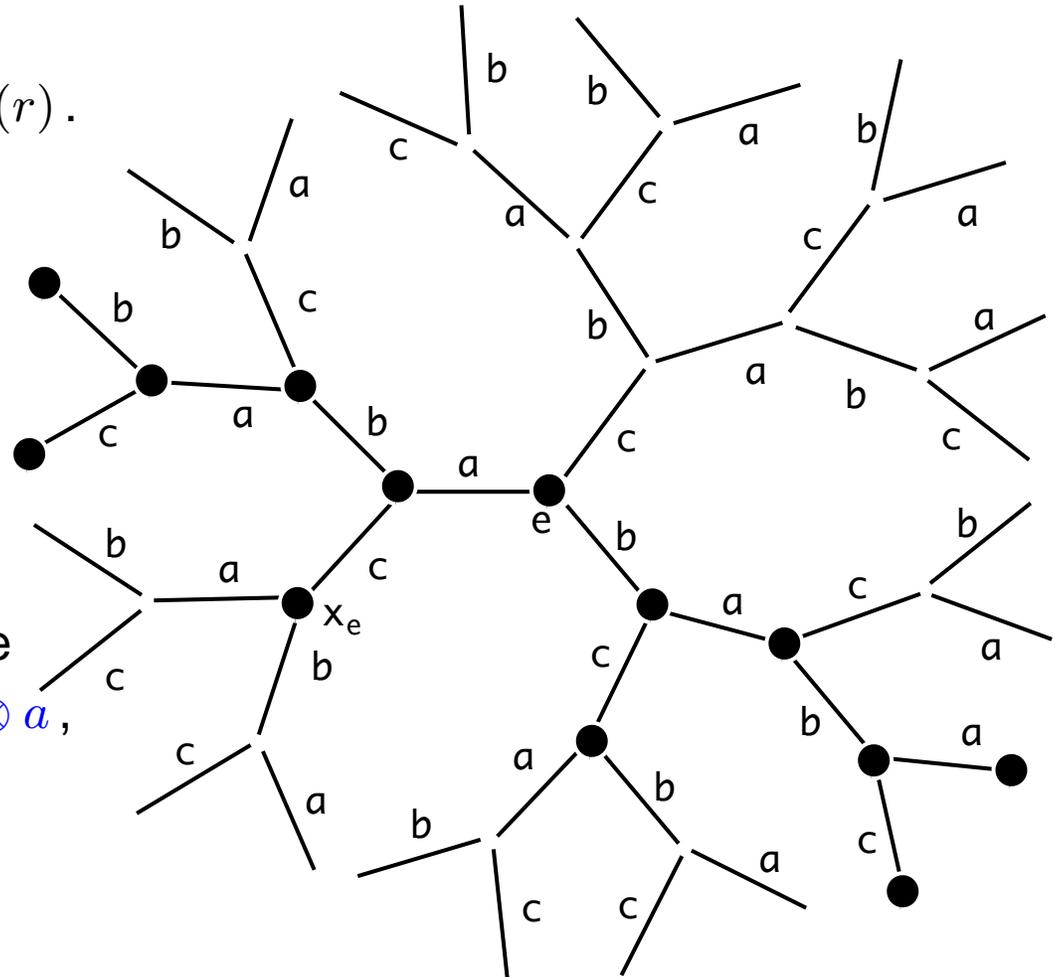
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For all $x_e \in \mathcal{T}_3$, let $\mathcal{H}_{x_e \otimes a}$ be the $U_\omega(C_1^l(r))$ -cyclic subsp. for $x_e \otimes a$,

Interpolating Matrices II

Delocalizing Matrices: For $0 \leq r \leq 1$ and $t = \sqrt{1 - r^2}$,

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Path counting argument:

$$\sum_{n \in \mathbb{Z}} |\langle x \otimes \tau | U^n x \otimes \tau \rangle|^2 < \infty \Rightarrow x \otimes \tau \in \text{a.c. spectral subspace of } U.$$

$$\langle x \otimes \tau | U^n x \otimes \tau \rangle = \sum_{\substack{y_j \in \mathcal{T}_3 \\ \sigma_j \in A_3}} \langle x \otimes \tau | U y_1 \otimes \sigma_1 \rangle \langle y_1 \otimes \sigma_1 | U y_2 \otimes \sigma_2 \rangle \cdots \langle y_{n-1} \otimes \sigma_{n-1} | U x \otimes \tau \rangle$$

Spin $|b\rangle$ or $|c\rangle$:

s.t. $x_e \mapsto x_e c$ or $x_e a$ whereas $x_o \mapsto x_o a$ or $x_o b$.

Only possibility for return in $2n$ steps $x \mapsto x a a a \dots a \rightsquigarrow \sum_{n \in \mathbb{N}} t^{2n} < \infty$.

Neighborhood of $C_{(a)(b)(c)} = \mathbb{I}$

Perturbation: $C = \mathbb{I} + E$, with $\|E\| \leq \epsilon$

$$C_{\tau\tau} = O(1), \quad C_{\tau\sigma} = O(\epsilon), \quad \tau \neq \sigma$$

Path counting argument to show $\langle x \otimes \tau | U_{\omega}^n(C) | x \otimes \tau \rangle \in l^2(\mathbb{Z})$ if ϵ small

Expansion \rightsquigarrow

$$\text{Path: } x = x \ y_1 \ y_2 \ y_3 \ \cdots \ y_{2n} \ \leftrightarrow \ \text{Weight: } |C_{\tau\sigma_1} C_{\sigma_1\sigma_2} \cdots C_{\sigma_{2n}\tau}|$$

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Lemma:

Weight of length $2n$ path from x to x contains at least n off diag. terms

Argument:

Strings with of m consecutive diag. elems cannot reduce one another
 \Rightarrow the need for enough off-diag. elems to ensure reduction.

Then: $\text{Weight} \leq \epsilon^n$, $\#\{\text{contrib. paths}\} \leq k^n$, $k \simeq 72 \Rightarrow$

$$\sum_{n \in \mathbb{N}} |\langle x \otimes \tau | U_{\omega}^n(C) x \otimes \tau \rangle|^2 \leq \sum_{n \in \mathbb{N}} (\epsilon k)^n < \infty, \text{ if } \epsilon > 0 \text{ small enough.}$$

Neighborhoods of $C_{(abc)}$ and $C_{(acb)}$

Theorem Let $\pi \in \{(abc), (acb)\}$, C_π and $U_\omega(C)$ be as above and $\mathbb{U} := \{|z| = 1\}$.

For all $\gamma > 0$, there exists $\delta > 0$, $K < \infty$, s.t. $\forall C \in U(3)$,
 $\|C - C_\pi\| < \delta \implies \forall x, y \in \mathcal{T}_3$ and $\forall \sigma, \tau \in A_3$

$$\mathbb{E} \left[\sup_{f \in C(\mathbb{U}), \|f\|_\infty \leq 1} |\langle x \otimes \tau | f(U_\omega(C)) y \otimes \sigma \rangle| \right] \leq K e^{-\gamma d(x,y)}$$

à la "Aizenman-Molchanov" '09 Hamza-J.-Stolz
Similar approach Asch, Bourget, J. 11', J. '12

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\implies **Spectral localization:**

$\sigma(U_\omega(C))$ is pure point, **a.s.** (Enss-Veselic '83)

\implies **"Exponential" Dynamical Localization:**

with $X x \otimes \tau = x x \otimes \tau$, ψ_0 cpct. support, and $p > 0$,

$$\sup_{n \in \mathbb{Z}} \|e^{p|X|} U_\omega^n(C) \psi_0\| < K_\omega, \text{ a.s.}$$

Remarks

- Generalizations to **homog. trees of coord. number $q \geq 3$** :

\exists **open sets** $\mathcal{L} \subset \mathbb{C}^q$, resp. $\mathcal{D} \subset \mathbb{C}^q$ s.t.

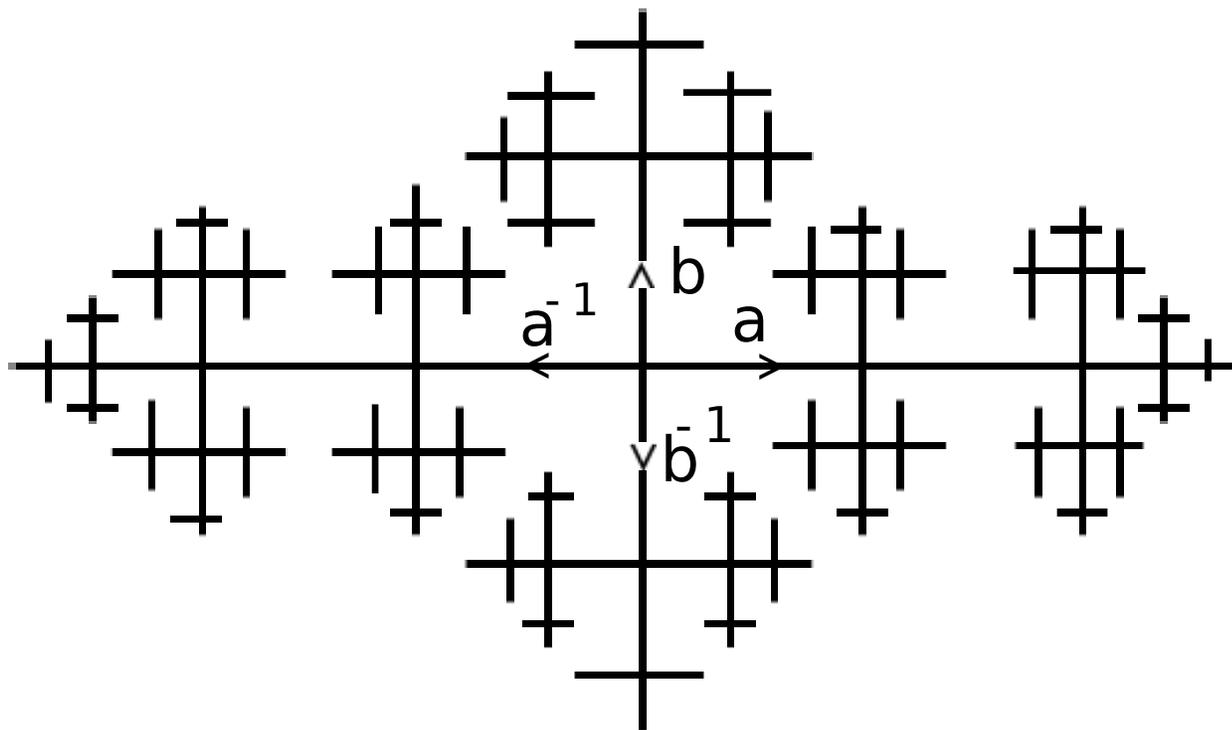
$C \in \mathcal{L} \Rightarrow \sigma(U_\omega(C))$ is **p.p.**, resp. $C \in \mathcal{D} \Rightarrow \sigma(U_\omega(C))$ is **a.c.**

- **RQW "analog" of weak disorder Anderson deloc. on trees** '94 Klein
- Spectral analysis for the **deterministic case ($q = 3$)** '13 J.-Marin

Highlights of RQW in \mathcal{T}_4

The tree \mathcal{T}_4 :

Hilbert space: $\mathcal{K}_4 = l^2(\mathcal{T}_4) \otimes \mathbb{C}^4$



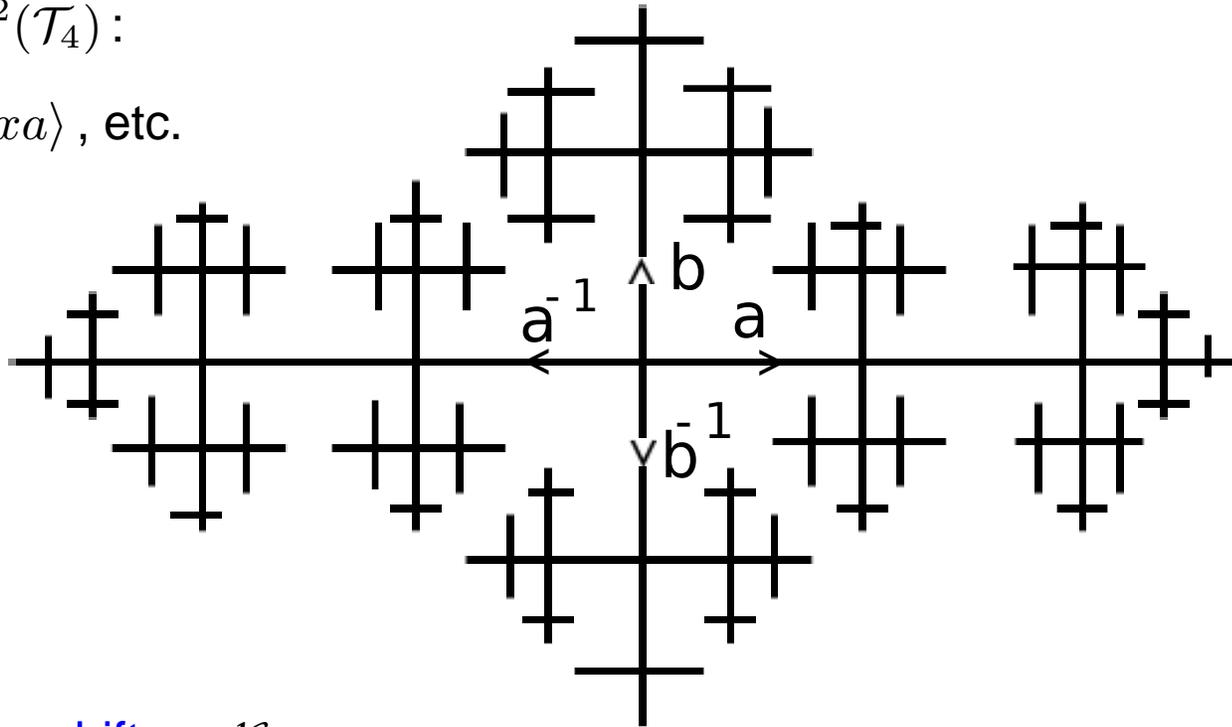
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Shift on $l^2(\mathcal{T}_4)$:

$S_a|x\rangle = |xa\rangle$, etc.



• Spin dep. shift on \mathcal{K}_4 :

$$S = S_a \otimes |a\rangle\langle a| + S_b \otimes |b\rangle\langle b| + S_{a^{-1}} \otimes |a^{-1}\rangle\langle a^{-1}| + S_{b^{-1}} \otimes |b^{-1}\rangle\langle b^{-1}|$$

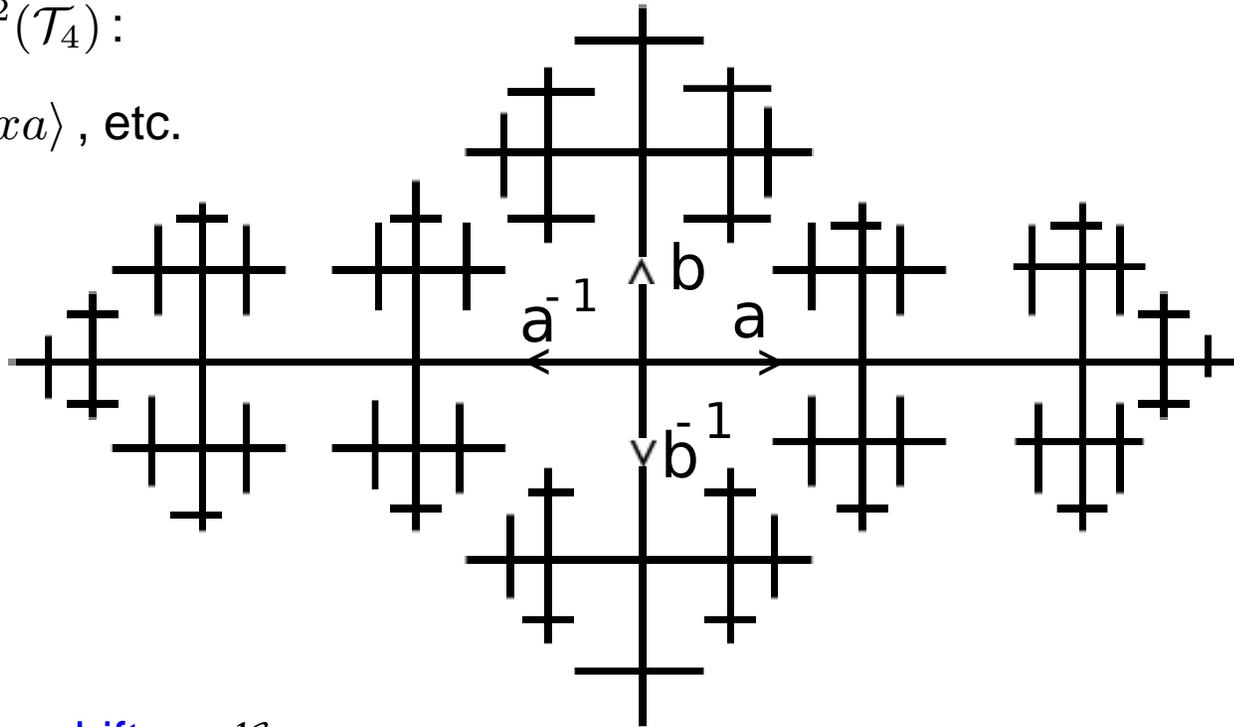
Highlights of RQW in \mathcal{T}_4

The tree \mathcal{T}_4 :

Hilbert space: $\mathcal{K}_4 = l^2(\mathcal{T}_4) \otimes \mathbb{C}^4$

Shift on $l^2(\mathcal{T}_4)$:

$S_a|x\rangle = |xa\rangle$, etc.



• Spin dep. shift on \mathcal{K}_4 :

$$S = S_a \otimes |a\rangle\langle a| + S_b \otimes |b\rangle\langle b| + S_{a^{-1}} \otimes |a^{-1}\rangle\langle a^{-1}| + S_{b^{-1}} \otimes |b^{-1}\rangle\langle b^{-1}|$$

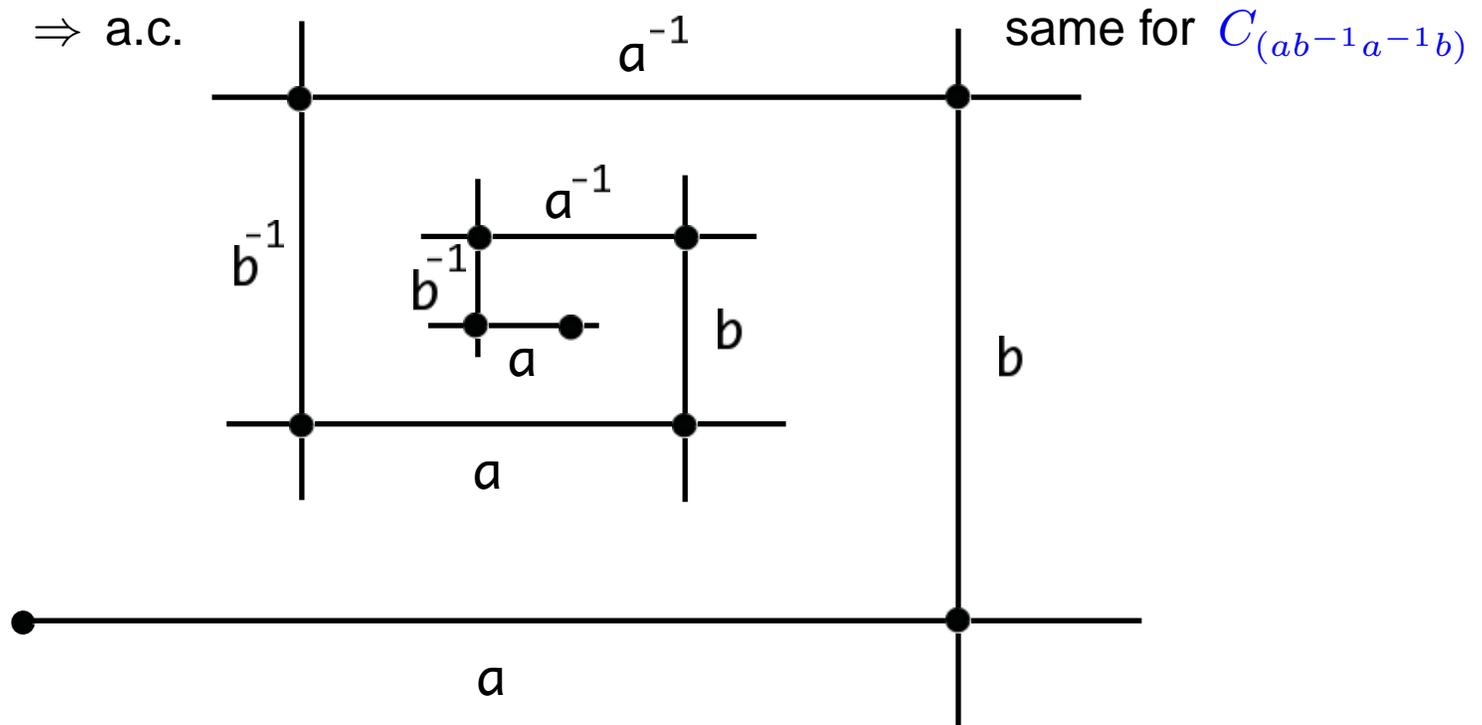
• Spin update: Via $C \in U(4)$

• RQW: $U_\omega(C) := \mathbb{D}(\omega)S(\mathbb{I} \otimes C)$

Highlights of RQW in \mathcal{T}_4

- $C_{(a)(b)(a^{-1})(b^{-1})} = \mathbb{I} \Rightarrow$ a.c. $U_\omega(\mathbb{I}) \simeq S.$

- $C_{(aba^{-1}b^{-1})} \Rightarrow$ a.c.



- $C_{(aa^{-1})(bb^{-1})} \Rightarrow$ p.p.

same for $C_{(abb^{-1}a^{-1})}$, etc...

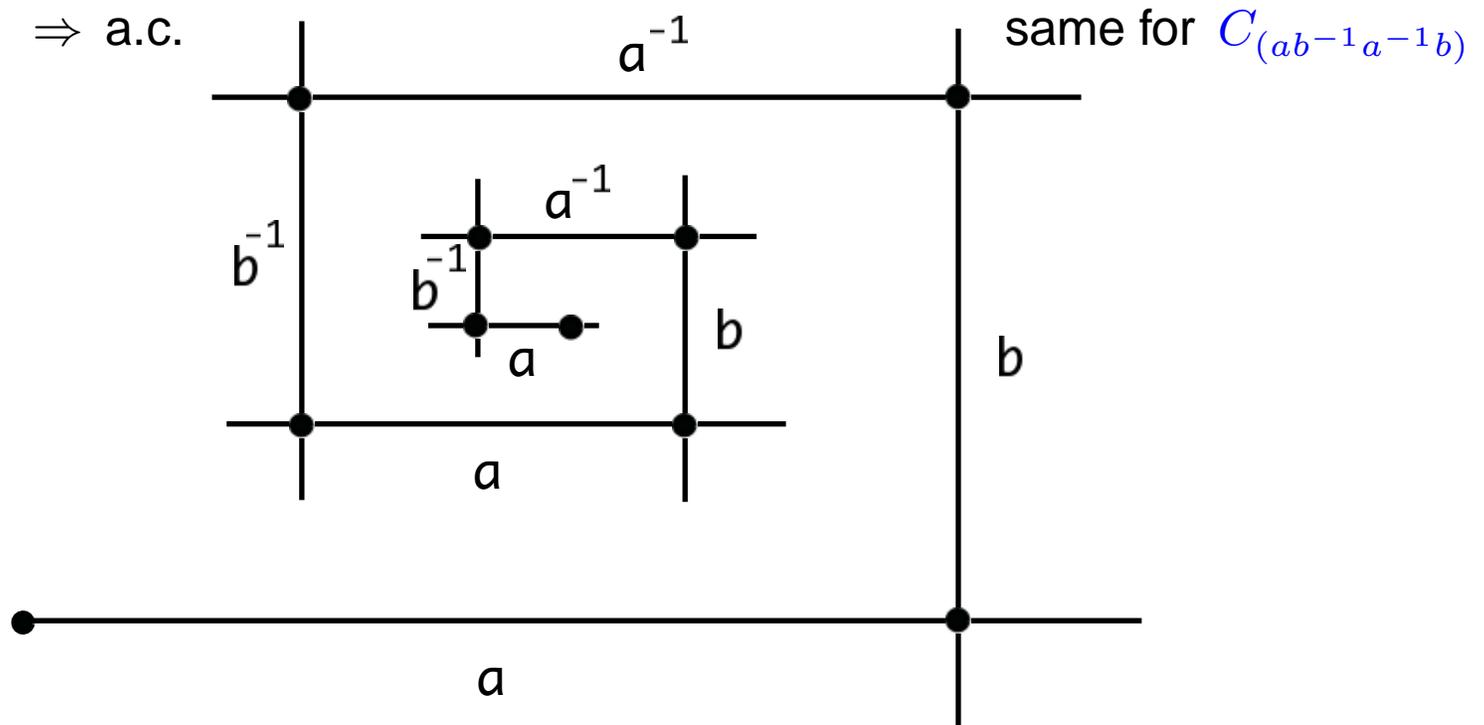
$$\mathcal{H}_x^a = \text{span}\{x \otimes a, xa^{-1} \otimes a^{-1}\}$$

$$\mathcal{H}_x^b = \text{span}\{x \otimes b, xb^{-1} \otimes b^{-1}\} \text{ invar. under } U_\omega(C_{(aa^{-1})(bb^{-1})}), \forall x \in \mathcal{T}_4.$$

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QRW in \mathcal{T}_4 : Highlights

Theorem:

