Quantum Repeated Measurements, Continuous Time Limit

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Plan

- **I) Discrete time Model: Quantum Repeated measurement**
  - H. Maasen: ergodic properties, purification properties.
  - L. Bouten, R. Van Handel: discrete quantum filtering
  - M. Bauer, D. Bernard, T. Benoist: large time behaviour, rate of convergence, quantum nondemolition measurement
  - P. Rouchon and all: control (direct collaboration with the serge Haroche team at LKB)
  - M. Merkli: quantum measurement of scattered particles
  - S. Attal, N. Guillotin-Plantard, C. Sabot: Central Limit Theorem for OQRW
- **II) Continuous Time model: Stochastic Schrödinger Equations, Stochastic Master Equations.**
  - Using linear stochastic master equations and change of measure (A.Barchielli, M.Gregoratti...)
  - Quantum Filtering Theory based on Quantum stochastic differential equation (V.P.Belavkin...)
  - Positive Operator Valued Measure and Instruments (E.B Davies...)
  - From discrete to continuous time model (adapting the results of S.Attal-Y.Pautrat...)

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- **IV) Estimation, Temperature**
  - Estimation: work in progress with P. Rouchon And H. Amini
  - Temperature (with I. Nechita and S. Attal)

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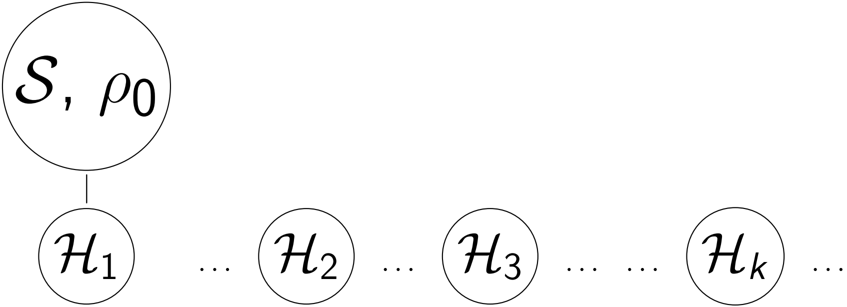
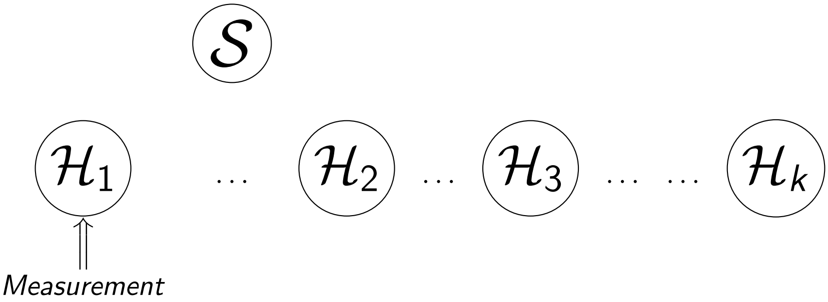
I) Discrete Time Model

QUANTUM REPEATED MEASUREMENT and DISCRETE QUANTUM TRAJECTORIES

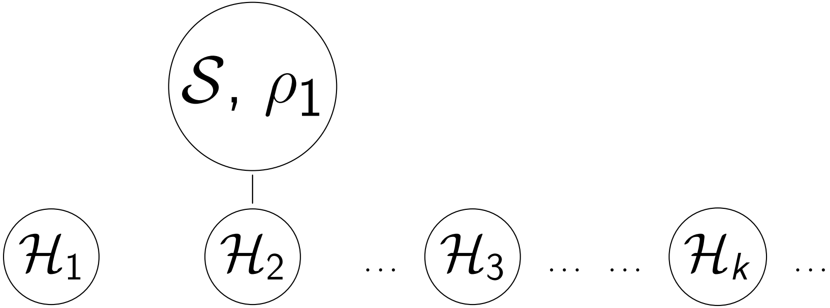
1st interaction



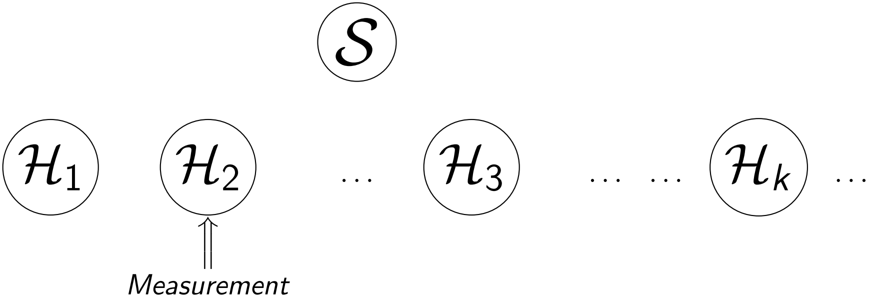
1st measurement



2nd interaction



2nd measurement



and so on $\Rightarrow$ discrete quantum trajectory (ρ_n)

Proposition

Let A be an observable of the form $A = \sum_{i=0}^{p} \lambda_i P_i$.

Then there exists a probability space $(\Omega, \mathcal{C}, P)$, *where the the discrete quantum trajectory* (ρ_k), *describing the quantum repeated measurement of A, is a Markov chain.*

More precisely if $\rho_k = \theta$ *is a state on* $\mathcal{H}_0$, *then* ρ_{k+1} *takes the values*

$$\frac{\mathcal{L}_i(\theta)}{Tr[\mathcal{L}_i(\theta)]}, \quad i \in \{0, \dots, p\}$$

where $\mathcal{L}_i(\theta) = Tr_{\mathcal{H}}[(I \otimes P_i)\ U(\theta \otimes \beta)U^\star\ (I \otimes P_i)]$. *Each state appears with probability*

$$p_i(\theta) = Tr[\mathcal{L}_i(\theta)].$$

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$$p_i(\theta) = Tr[\mathcal{L}_i(\theta)].$$

- The previous result can be summarized by writing the following evolution equation:

$$\rho_{k+1} = \sum_{i=0}^{p} \frac{\mathcal{L}_i(\rho_k)}{Tr[\mathcal{L}_i(\rho_k)]} \mathbf{1}_i^{k+1},$$

with $\mathcal{L}_i(\theta) = \mathrm{Tr}_{\mathcal{H}}[(I \otimes P_i)\ U(\theta \otimes \beta)U^\star\ (I \otimes P_i)]$.

- **Remark**: The operator U depends on the time interaction τ

$$U = e^{-i\tau H_{tot}}.$$

Questions

What gives the limit τ goes to 0?

What is the limit evolution?

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II) Stochastic Master Equations Continuous Quantum Trajectories

- Let us start with model where only one measurement apparatus is concerned
- Evolution of $\mathcal{H}_0$ without measurement : Master Equation in Lindblad form

$$d\rho_t = L(\rho_t)dt.$$

- Effect of measurement = perturbation of this ode under the form of stochastic differential equations

Stochastic Master Equations

- **Diffusive Equation**

$$d\rho_t = L(\rho_t)dt + [C\rho_t + \rho_t C^\star - Tr[\rho_t(C + C^\star)]\rho_t]dW_t$$

1. The process (W_t) is a standard Brownian motion.
2. C is an arbitrary operator appearing in the Lindblad operator, L.

- Often this equation appears on the following form

$$\begin{aligned} d\rho_t &= L(\rho_t)dt \\ &\quad +[C\rho_t + \rho_t C^\star - Tr[\rho_t(C + C^\star)]\rho_t](dy_t - \mathrm{Tr}[\rho_t(C + C^\star)]dt), \end{aligned}$$

where

$$dy_t = dW_t + \mathrm{Tr}[\rho_t(C + C^\star)]dt$$

- The process (y_t) represents the measurement process recorded by the measurement apparatus (Homodyne/Heterodyne detection in Quantum Optics).

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- **Jump Equation**

$$d\rho_t = L(\rho_t)dt + \left[\frac{\mathcal{J}(\rho_t)}{Tr[\mathcal{J}(\rho_t)]} - \rho_t\right]\left(d\tilde{N}_t - Tr\big[\mathcal{J}(\rho_t)\big]\,dt\right)$$

1. The process $(\tilde{N}_t)$ is a counting process of stochastic intensity

$$t \longrightarrow \int_0^t Tr[\mathcal{J}(\rho_s)]ds.$$

2. $\mathcal{J}(\rho) = C\rho C^\star.$

- The process $(\tilde{N}_t)$ represents the number of photon detected up to time t by a photo detector.

- First fact: $d\mathbb{E}[\rho(t)] = L(\mathbb{E}[\rho(t)])dt$. That is $(\mathbb{E}[\rho(t)])$ reproduces the solution of the Lindblad master equation
- In the previous cases if

$$L(\rho) = -i[H,\rho] - \frac{1}{2}\{C^\star C,\rho\} + C\rho C^\star$$

and if at time 0 $\rho_0 = |\psi_0\rangle\langle\psi_0|$ then there exists ψ_t such that

$$\rho(t) = |\psi_t\rangle\langle\psi_t|, \forall t$$

The equation satisfied by ψ_t is called a Stochastic Schrödinger Equation

- In general we have $\rho(t) = \rho^*(t)$ and $\mathrm{Tr}[\rho(t)] = 1$, then if there is a solution and if the initial condition is a density matrix then the solution is self-adjoint and of trace 1. What is very difficult to show is that the solution is positive.

Existence and uniqueness: the diffusive case

Theorem

Let $(\Omega, \mathcal{F}, P)$ a probability space where (W_t) is a standard Brownian motion.

the equation

$$\begin{aligned} d\rho_t &= L(\rho_t)dt \\ &\quad + \Big[C\rho_t + \rho_t C^\star - Tr[\rho_t(C + C^\star)]\rho_t\Big]dW_t \end{aligned}$$

admits a unique solution (ρ_t) with values in the set of states of $\mathcal{H}_0$.

- Non Lipschitz coefficients

  One can use a truncature method

- We show that this equation preserves the property of being a state.
- Here we can use the approximation procedure to show the positivity of the solution

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Processus de comptage $(\tilde{N}_t)$?

- Jump equation

$$d\rho_t = L(\rho_t)dt + \left[\frac{\mathcal{J}(\rho_t)}{Tr[\mathcal{J}(\rho_t)]} - \rho_t\right](d\tilde{N}_t - Tr[\mathcal{J}(\rho_t)]dt)$$

Recall: $(\tilde{N}_t)$ is a counting process of intensity $\int_0^t Tr[\mathcal{J}(\rho_s)]ds$.

Questions

What is the meaning of this equation?

How can we define $(\tilde{N}_t)$ and (ρ_t)?

Process-solution de $d\rho_t = L(\rho_t)dt + \left[\frac{\mathcal{J}(\rho_t)}{Tr[\mathcal{J}(\rho_t)]} - \rho_t\right](d\tilde{N}_t - Tr[\mathcal{J}(\rho_t)]dt)$

Definition

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ *be a probability space. A* **process-solution** *of the jump equation is a couple* $(\rho_t, \tilde{N}_t)$ *such that*

$$\begin{aligned}\rho_t &= \rho_0 + \int_0^t \left[L(\rho_{s-}) - \mathcal{J}(\rho_{s-}) + Tr[\mathcal{J}(\rho_{s-})]\rho_{s-}\right] ds \\ &\quad + \int_0^t \left[\frac{\mathcal{J}(\rho_{s-})}{Tr[\mathcal{J}(\rho_{s-})]} - \rho_{s-}\right] d\tilde{N}_s \quad a.s\end{aligned}$$

and such that

$$\tilde{N}_t - \int_0^t Tr[\mathcal{J}(\rho_{s-})]ds$$

is a $\mathcal{F}_t$*-martingale.*

Existence and uniqueness

Theorem

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a probability space equipped with a random Poisson measure μ on $\mathbb{R}^+ \times \mathbb{R}$ whose the intensity measure is the Lebesgue measure $ds \otimes dx$. The SDE

$$\rho_t = \rho_0 + \int_0^t \Big[L(\rho_{s-}) - \mathcal{J}(\rho_{s-}) + Tr[\mathcal{J}(\rho_{s-})]\rho_{s-} \Big] ds$$
$$+ \int_0^t \int_{\mathbb{R}} \left[\frac{\mathcal{J}(\rho_{s-})}{Tr[\mathcal{J}(\rho_{s-})]} - \rho_{s-} \right] \mathbf{1}_{0<x<tr[\mathcal{J}(\rho_{s-})]}\, \mu(ds, dx)$$

admits a unique solution (ρ_t). The process $(\tilde{N}_t)$ defined by

$$\tilde{N}_t = \int_0^t \int_{\mathbb{R}} \mathbf{1}_{0<x<tr[\mathcal{J}(\rho_{s-})]}\, \mu(ds, dx)$$

is a counting process of intensity $\int_0^t Tr[\mathcal{J}(\rho_{s-})] ds$.

Generalisation

One can generalise

$$\begin{aligned}\rho_t &= \rho_0 + \int L(\rho_{s-})ds + \sum_{i=0}^{l} \int_0^t h_i(\rho_{s-})dW_i(s) \\ &\quad + \sum_{i=0}^{q} \int_0^t \int_{\mathbb{R}} g_i(\rho_{s-}) \mathbf{1}_{0<x<v_i(\rho_{s-})} \left[\mu_i(dx, ds) - dxds\right],\end{aligned}$$

where $(W_t = (W_0(t), \dots, W_p(t))$ are a p-dimensional Brownian motion and μ_i are $p+1$ random measure of intensity $ds \otimes dx$. All the processes are independent.

Remark The functions h_i et g_i are given by

$$\begin{aligned}h_i(\rho) &= C_i\rho + \rho C_i^\star - Tr[\rho(C_i + C_i^\star)]\rho \\ g_i(\rho) &= \frac{\mathcal{J}_i(\rho)}{Tr[\mathcal{J}_i(\rho)]} - \rho\end{aligned}$$

III) Convergence Result

FROM DISCRETE TO CONTINUOUS QUANTUM TRAJECTORIES

Back to the discrete setup

- Setup
  1. Case $\mathcal{H}_0 = \mathcal{H} = \mathbb{C}^2$
  2. An observable of $\mathcal{H}$ is of the form $A = \lambda_0 P_0 + \lambda_1 P_1$.
  3. The discrete stochastic Schrödinger equation

$$\rho_{k+1} = \frac{\mathcal{L}_0(\rho_k)}{p_0(\rho_k)} \mathbf{1}_0^{k+1} + \frac{\mathcal{L}_1(\rho_k)}{p_1(\rho_k)} \mathbf{1}_1^{k+1}.$$

- Let us introduce $X_{k+1} = \dfrac{\mathbf{1}_1^{k+1} - p_1(\rho_k)}{\sqrt{p_0(\rho_k)p_1(\rho_k)}}$.
- In terms of X_{k+1}, we get

$$\rho_{k+1} = \mathcal{L}_0(\rho_k) + \mathcal{L}_1(\rho_k) + \left[-\sqrt{\frac{p_0}{p_1}} \mathcal{L}_0(\rho_k) + \sqrt{\frac{p_1}{p_0}} \mathcal{L}_1(\rho_k) \right] X_{k+1}.$$

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Asymptotic assumptions

- Recall that (ρ_k) is defined through the quantity

$$\mathcal{L}_i(\rho) = \mathrm{Tr}_{\mathcal{H}}[\,(I \otimes P_i)\;\; U(n)(\rho \otimes \beta)U^\star(n)\;\; (I \otimes P_i)\,]$$

$$U(n) = e^{i\frac{1}{n}H_{tot}}$$

Naturally the asymptotic assumptions are going to appear in $U(n)$.

- Now, we fix a basis $\{\Omega_0, \Omega_1\}$
The reference state of the chain will be $\beta = |\Omega_0\rangle\langle\Omega_0|$.

Expression de U(n)

- Let us write $U(n)$ as

$$U(n) = \begin{pmatrix} U_0^0(n) & U_0^1(n) \\ U_1^0(n) & U_1^1(n) \end{pmatrix}$$

where the $U_{ij}(n)$ are operators $\mathcal{H}_0$.

- **S. Attal-Y. Pautrat**: From repeated to continuous quantum interactions, "*Annales Henri Poincaré*"
  In the previous article, the authors gives a precise description of the asymptotic conditions that we need to impose to $U_{ij}(n) \Longrightarrow$ in order to obtain a non-trivial limit for the quantum repeated interactions model (interms of quantum stochastic calculus).
- In our context, we naturally adopt their conditions and we need

$$\begin{aligned} U_0^0(n) &= I + \frac{1}{n}\left(-iH_0 - \frac{1}{2}C^\star C\right) + \circ\left(\frac{1}{n}\right) \\ U_1^0(n) &= \frac{1}{\sqrt{n}}C + \circ\left(\frac{1}{n}\right) \end{aligned}$$

- If A is **diagonal** in $\{\Omega_0, \Omega_1\}$. For example $A = 1 \times |\Omega_0\rangle\langle\Omega_0| + 0 \times |\Omega_1\rangle\langle\Omega_1|$, then we have

$$\mathcal{L}_0(\rho) = U_0^0 \rho (U_0^0)^\star = \rho + \frac{1}{n}\left[(-iH_0 - \frac{1}{2}C^\star C)\rho + \rho(-iH_0 - \frac{1}{2}C^\star C)\right]$$

$$\mathcal{L}_1(\rho) = U_1^0 \rho (U_1^0)^\star = \frac{1}{n} C \rho C^\star$$

The **transition probabilities** satisfies

$$\begin{aligned} p_0(\rho_k) &= 1 - \frac{1}{n} Tr[\mathcal{J}(\rho_k)] + \circ\left(\frac{1}{n}\right) \\ p_1(\rho_k) &= \frac{1}{n} Tr[\mathcal{J}(\rho_k)] + \circ\left(\frac{1}{n}\right). \end{aligned}$$

Limit evolution in the case of a diagonal A

- If A is **diagonal** in $\{\Omega_0, \Omega_1\}$. then we have

$$\begin{aligned} \rho_{k+1} &= \rho_k + \frac{1}{n}[L(\rho_k) + \circ(1)] \\ &\quad + \left(\frac{\mathcal{J}(\rho_k)}{Tr[\mathcal{J}(\rho_k)]} - \rho_k + \circ(1) \right) (\mathbf{1}_1^{k+1} - p_1(\rho_k)). \end{aligned}$$

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In case of a non diagonal A

- If $A = \lambda_0 P_0 + \lambda_1 P_1$ de $\mathcal{H}$ **is not diagonal** in $\{\Omega_0, \Omega_1\}$. For example $A = |\Omega_0\rangle\langle\Omega_1| + |\Omega_1\rangle\langle\Omega_0|$ we get the following asymptotic expression

$$\begin{aligned} \mathcal{L}_0(\rho) &= \frac{1}{2}\left(U_0^0\rho(U_0^0)^\star + U_0^0\rho(U_1^0)^\star + U_1^0\rho(U_0^0)^\star + U_1^0\rho(U_1^0)^\star\right) && (1) \\ &= \frac{1}{2}\left(\rho + \frac{1}{\sqrt{n}}(C\rho + \rho C^\star) + \frac{1}{n}L(\rho)\right) && (2) \end{aligned}$$

$$\mathcal{L}_1(\rho) = \frac{1}{2}\left(U_0^0\rho(U_0^0)^\star - U_0^0\rho(U_1^0)^\star - U_1^0\rho(U_0^0)^\star + U_1^0\rho(U_1^0)^\star\right)$$

- Here the **probabilities** are

$$\begin{aligned} p_0(\rho_k) &= \frac{1}{2} + \frac{1}{\sqrt{n}}\left[Tr[\rho_k(C + C^\star)] + \circ(1)\right] \\ p_1(\rho_k) &= 1 - p_0(\rho_k). \end{aligned}$$

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$$\rho_{k+1} = \rho_k + \frac{1}{n}[L(\rho_k) + \circ(1)] + \frac{1}{\sqrt{n}}\Big[C\rho_k + \rho_k C^\star - Tr[\rho_k(C + C^\star)]\rho_k + \circ(1)\Big]X_{k+1}.$$

Here the **probabilities** are

$$\begin{aligned} p_0(\rho_k) &= \xi + \frac{1}{\sqrt{n}}\nu\Big[Tr[\rho_k(C + C^\star)] + \circ(1)\Big] \\ p_1(\rho_k) &= 1 - p_0(\rho_k). \end{aligned}$$

Convergence to the diffusive case

- From the previous description, we put

$$\rho_{[nt]} = \rho_0 + \sum_{k=0}^{[nt]-1} \frac{1}{n}[L(\rho_k) + \circ(1)] + \sum_{k=0}^{[nt]-1} \frac{1}{\sqrt{n}}[\mathcal{H}(\rho_k) + \circ(1)]X_{k+1}.$$

- Putting

$$\rho_n(t) = \rho_{[nt]}, \qquad V_n(t) = \frac{[nt]}{n}, \qquad W_n(t) = \frac{1}{\sqrt{n}} \sum_{k=0}^{[nt]-1} X_{k+1}.$$

- We have that $(\rho_n(t))$ satisfies

$$\rho_n(t) = \rho_0 + \int_0^t L(\rho_n(s-))dV_n(s) + \int_0^t \mathcal{H}(\rho_n(s-))dW_n(s) + \varepsilon_n(t).$$

Convergence - Kurtz-Protter

Theorem

The process $(W_n(t), V_n(t), \varepsilon_n(t))$ *converge in distribution to* $(W_t, V_t, 0)$ *where* (W_t) *is a standard Brownian motion and* $V_t = t$ *for all* t.
Moreover, we have

$$\sup_n \mathbf{E}\Big[[W_n(t), W_n(t)]\Big] < \infty$$

Then the process $(\rho_n(t))$ *satisfying*

$$\rho_n(t) = \rho_0 + \int_0^t L(\rho_n(s-))dV_n(s) + \int_0^t \mathcal{H}(\rho_n(s-))dW_n(s) + \varepsilon_n(t)$$

converge in ditribution to (ρ_t) *the unique solution of*

$$\rho_t = \rho_0 + \int_0^t L(\rho_s)ds + \int_0^t \mathcal{H}(\rho_s)dW_s.$$

The jump case

- Again

$$\rho_{[nt]} = \rho_0 + \sum_{k=0}^{[nt]-1} \frac{1}{n}\Big[L(\rho_k) - \mathcal{J}(\rho_k) + Tr[\mathcal{J}(\rho_k)]\rho_k + \circ(1)\Big] + \sum_{k=0}^{[nt]-1} \left(\frac{\mathcal{J}(\rho_k)}{Tr[\mathcal{J}(\rho_k)]} - \rho_k + \circ(1)\right)\mathbf{1}_1^{k+1}.$$

- Again, we put

$$\rho_n(t) = \rho_{[nt]}\,, \qquad V_n(t) = \frac{[nt]}{n}\,, \qquad N_n(t) = \sum_{k=0}^{[nt]-1} \mathbf{1}_1^{k+1}.$$

- We get the discrete SDE

$$\rho_n(t) = \rho_0 + \int_0^t \Theta(\rho_n(s-))dV_n(s) + \int_0^t \Phi(\rho_n(s-))dN_n(s) + \varepsilon_n(t).$$

Convergence

- Kurtz-Protter?
- In the jump case, **we can not directly show that** $(N_n(t))$ converge in distribution to the process $(\tilde{N}_t)$.
- Method:
  1. Coupling.
  2. Comparison with a Euler scheme.

Theorem

The process $(\rho_n(t))$ defined from the quantum repeated measurement of a diagonal observable converge in distribution to (ρ_t) solution of the jump equation.

- Kurtz-Protter?
- In the jump case, **we can not directly show that** $(N_n(t))$ converge in distribution to the process $(\tilde{N}_t)$.
- Method:
  1. Coupling.
  2. Comparison with a Euler scheme.

Theorem

The process $(\rho_n(t))$ defined from the quantum repeated measurement of a diagonal observable converge in distribution to (ρ_t) solution of the jump equation.

General case

- How can we compare the two method?
- How can we show that

$$\rho_{k+1} = \sum_{i=0}^{p} \frac{\mathcal{L}_i(\rho_k)}{Tr[\mathcal{L}_i(\rho_k)]} \mathbf{1}_i^{k+1}$$

converges to

$$\begin{aligned} \rho_t &= \rho_0 + \int L(\rho_{s-})ds + \sum_{i=0}^{l} \int_0^t h_i(\rho_{s-})dW_i(s) \\ &+ \sum_{i=0}^{q} \int_0^t \int_{\mathbb{R}} g_i(\rho_{s-}) \mathbf{1}_{0<x<v_i(\rho_{s-})} \left[\mu_i(dx, ds) - dxds\right]. \end{aligned}$$

- **Martingale problem.**

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- **Martingale problem**.

General case

- The discrete model can be describe by the transition Kernel

$$\Pi_n(\rho,\mu)=\sum_{i=0}^{p} p^i(\rho)\delta_{\mathcal{L}_i^{(n)}(\rho)/\mathrm{Tr}[\mathcal{L}_i^{(n)}(\rho)]}(\mu), \quad (3)$$

- We can then define the Markov generator

$$\begin{aligned}\mathcal{A}_n f(\rho) &= n\int (f(\mu)-f(\rho))\Pi_n(\rho,d\mu)\\ &= n\sum_{i=0}^{p}\Big(f(\mathcal{L}_i^{(n)}(\rho)/\mathrm{Tr}[\mathcal{L}_i^{(n)}(\rho)])-f(\rho)\Big)p^i(\rho). \quad (4)\end{aligned}$$

- Defining $\rho_n(t)=\rho_{[nt]}$ and $\mathcal{F}_t^n=\sigma(\rho_n(s),s\leqslant t)$

$$f(\rho_n(k/n))-f(\rho_0)-\sum_{j=0}^{k-1}\frac{1}{n}\mathcal{A}_n f(\rho_n(j/n)) \quad (5)$$

is a $(\mathcal{F}_{k/n}^n)$ martingale

General case

- You compute the limit of $\mathcal{A}_n$ denoted by $\mathcal{A}$
- In distribution, there exists a unique Markov process $\rho(t)$ such that

$$f(\rho(t) - f(\rho(0)) - \int_0^t \mathcal{A}f(\rho(s))ds$$

is a martingale with respect to the natural filtration of $(\rho(t))$

- After identifying $\mathcal{A}$, one can show that this Markov generator is the same as the one of the solution of the generalization of the stochastic master equation.
- This gives the expected convergence in distribution

IV) Estimation, Temperature

Work in progress

- The problem of estimation concerns similar models where we do not know the initial state. Nevertheless we have access to the results of the measurement (for example the value 0 or 1).

- If ρ design the true initial state (that we do not know), we know that 0 appears with probability $p_0(\rho)$ and 1 with probability $p_1(\rho)$.
- Now let $\tilde{\rho}$ an arbitrary state, conditionally to the result of the measurement, we put

$$\tilde{\rho}_1(i) = \frac{\mathcal{L}_i(\tilde{\rho})}{Tr[\mathcal{L}_i(\tilde{\rho})]}$$

depending on wether we observe 0 or 1.

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Estimation

Work in progress

- What can we expect?
- We can expect that the distance between $\tilde{\rho}_1$ and ρ_1 is smaller than the one between $\tilde{\rho}$ and ρ.
- Roughly speaking this means that knowing the result of the measurement allows us to estimate the true quantum trajectory.
- Within the previous procedure we can produce a random sequence $\tilde{\rho}_k$ whose transition probability are given by the one of the "true" quantum trajectory .

Work in progress

- Within the previous procedure we can produce a random sequence $\tilde{\rho}_k$ whose transition probability are given by the one of the "true" quantum trajectory.
- In order to evaluate if the "true" and the "estimate" quantum trajectory get closer we use the fidelity distance

$$F(\rho,\mu) = Tr[\sqrt{\sqrt{\rho}\mu\sqrt{\rho}}]^2$$

- We have $F(\rho,\mu) = F(\mu,\rho)$ and if for example $\rho = |\psi\rangle\langle\psi|$

$$F(\rho,\mu) = Tr[\rho\mu]$$

- $F(\rho,\mu) = 1$ if and only if $\rho = \mu$.

Work in progress

- We have the following result

$$\mathbb{E}[F(\tilde{\rho}_{k+1}, \rho_{k+1})|(\tilde{\rho}_k, \rho_k)] \geqslant F(\tilde{\rho}_k, \rho_k),$$

- This means that $F(\tilde{\rho}_k, \rho_k)$ is a sub-martingale.
- It is not a trivial result since there exist distance where this property is not satisfied.
- Problem: What do we have to impose on the system to have $\lim_{k\to\infty} F(\tilde{\rho}_k, \rho_k) = 1$
- Similar result for continuous time models (the discrete approach is really useful to show the sub-martingale result).

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Work in progress

- Similar result for continuous time models.

$$\begin{aligned} d\rho(t) &= L(\rho(t-))dt + \sum_{i=0}^{p} H_i(\rho(t-))(dy_i(t) - \mathrm{Tr}[(C_i + C_i^*)\rho(t-)]dt) \\ &\quad + \sum_{i=p+1}^{n} \left(\frac{J_i(\rho(t-))}{v_i(\rho(t-))} - \rho(t-) \right) (dN_i(t) - v_i(\rho(t-))dt). \\ \rho(0) &= \rho_0 \end{aligned}$$

$$\begin{aligned} d\tilde{\rho}(t) &= L(\tilde{\rho}(t-))dt + \sum_{i=0}^{p} H_i(\tilde{\rho}(t-))(dy_i(t) - \mathrm{Tr}[(C_i + C_i^*)\tilde{\rho}(t-)]dt) \\ &\quad + \sum_{i=p+1}^{n} \left(\frac{J_i(\tilde{\rho}(t-))}{v_i(\tilde{\rho}(t-))} - \tilde{\rho}(t-) \right) (dN_i(t) - v_i(\tilde{\rho}(t-))dt). \\ \tilde{\rho}(0) &= \tilde{\rho}_0, \end{aligned}$$

Work in progress

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Work in progress

- The fidelity is still a sub-martingale
- Almost impossible to estimate the term

$$dF(\tilde{\rho}(t), \rho(t)) = d\mathrm{Tr}[\sqrt{\sqrt{\tilde{\rho}(t)}\rho(t)\sqrt{\tilde{\rho}(t)}}]^2$$

- Use of the sub-martingale property for the discrete time result and the convergence result

Temperature

Temperature

- The state of the environment $\beta = |\Omega_0\rangle\langle\Omega_0|$ represents the vacuum state. Considering such a state is crucial in the approach of Attal-Pautrat.
- In the work of Attal-Joye, they consider a modelization of a heat bath by taking

$$\beta = \frac{e^{-\beta H}}{Z} = \begin{pmatrix} \beta_0 & 0 \\ 0 & \beta_1 \end{pmatrix},$$

  where β is the inverse of a temperature. this is the usual Gibbs state. Using a GNS representation and adapting the work of Attal-Pautrat they manage to derive Quantum Langevin Equation for heat bath.

Temperature

- What are the limit equation in the context of QRM.
- Following the guideline of Attal-Joye we use the GNS representation and we adapt the general result (with multiple noise). Surprisingly (for me) no jumping processes remain.
- For example in the case of two results, in the case of the diagonal observable, the limit equations is deterministic: just the Lindblad master equation (with temperature parameter)
- In the case of non diagonal the Wiener process remains

Temperature

- Alternative: from a physical point of view, you can consider that the Gibbs state $\begin{pmatrix} \beta_0 & 0 \\ 0 & \beta_1 \end{pmatrix}$, is either

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

with probability β_0 or

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

with probability β_1.

- One can consider the evolution without measurement, the random aspect disappears at the limit and we recover the Lindblad equation (as if the random aspect is averaged)
- But with measurement, in the diagonal case it appears two different jumps and in the non diagonal case two different Wiener processes.

THANK YOU