Indirect Quantum Non-demolition Measurement: continuous time model

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Plan

- **I) Discrete Time Model.**
  - Talk of T. Benoist
- **II) Continuous Time Model.**
  - Work in collaboration with Tristan Benoist (LPT ENS-Paris)
  - Large time behavior (QND)
  - Estimation
- **III) From discrete to continuous**

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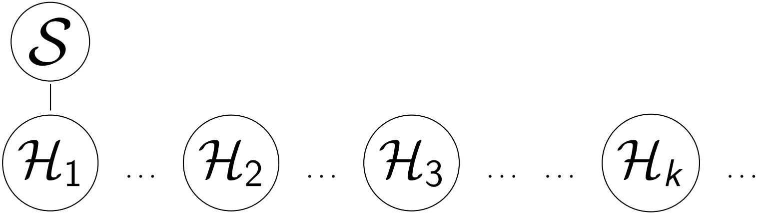
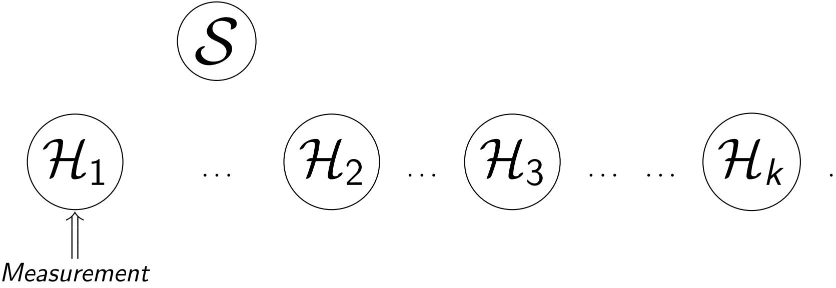
I) Discrete Time model

"Previously on Quantum Repeated Measurement and on Quantum nondemolition Measurement"

interaction + measurement / interaction + measurement

- 

- 

II) Continuous Time Model

Stochastic Master Equations

- The setup is a quantum system $\mathcal{S}$ wich is continuously monitored by a measurement apparatus

- In quantum Optics: Homodyne/Heterodyne detection, direct photodetection.

Quantum Non Demolition

- The quantities $q_\alpha(t) = \mathrm{Tr}(\rho(t-)|\alpha\rangle\langle\alpha|)$ satisfy

$$dq_\alpha(t) = q_\alpha(t-)\Bigg[\sum_{i=0}^{p}\big(r(i|\alpha) - \langle r_i(t-)\rangle\big)dW_i(s) + \sum_{i=p+1}^{n}\left(\frac{\theta(i|\alpha)}{\langle\theta_i(t-)\rangle} - 1\right)\Big(dN_i(t) - \langle\theta_i(t-)\rangle dt\Big)\Bigg] \quad (1)$$

$$\text{with} \quad r(i|\alpha) = c(i|\alpha) + \overline{c(i|\alpha)}, \quad \theta(i|\alpha) = |c(i|\alpha)|^2$$
$$\text{and} \quad \langle r_i(t)\rangle = \sum_\alpha q_\alpha(t) r(i|\alpha), \quad \langle\theta_i(t)\rangle = \sum_\alpha q_\alpha(t)\theta(i|\alpha) \quad (2)$$

Doleans equation

We choose a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ such that $W_i(t)$ and $N_i(t) - \int_0^t v_i(\rho(s))ds$ are $(\mathcal{F}_t)$ martingales.

Theorem

$(q_\alpha(t))$ are martingales, i.e, $\mathbb{E}[q_\alpha(t)|\mathcal{F}_s] = q_\alpha(s)$, $\forall s \leqslant t$. They satisfy

$$q_\alpha(t) = q_0(\alpha) \times \exp\Bigg[\sum_{i=0}^{p}\Bigg(\int_0^t \big(r(i|\alpha) - \langle r_i(s-)\rangle\big)\,dW_i(s) - \frac{1}{2}\int_0^t \big(r(i|\alpha) - \langle r_i(s-)\rangle\big)^2 ds\Bigg) + \sum_{i=p+1}^{n}\Bigg(\int_0^t \ln\left(\frac{\theta(i|\alpha)}{\langle\theta_j(s-)\rangle}\right) d\tilde{N}_i(s) - \int_0^t \big(\theta(i|\alpha) - \langle\theta_j(s-)\rangle\big)\,ds\Bigg)\Bigg]. \quad (3)$$

$$q_\alpha(t) =$$

$$= q_\alpha(0) \times \exp\left[\sum_{i=0}^{p}\left(\int_0^t \big(r(i|\alpha) - \langle r_i(s-)\rangle\big)\,dW_i(s) - \frac{1}{2}\int_0^t \big(r(i|\alpha) - \langle r_i(s-)\rangle\big)^2 ds\right)\right]$$

$$\times \prod_{i=p+1}^{n}\prod_{s\leqslant t}\left(1 + \left(\frac{\theta(i|\alpha)}{\langle\theta_j(s-)\rangle} - 1\right)\Delta\hat{N}_i(t)\right) \times \exp\left[-\int_0^t (\theta(i|\alpha) - \langle\theta_j(s-)\rangle))ds\right]$$

$$= q_\alpha(0)\times\exp\left[\sum_{i=0}^{p}\left(\int_0^t \big(r(i|\alpha) - \langle r_i(s-)\rangle\big)\,dW_i(s) - \frac{1}{2}\int_0^t \big(r(i|\alpha) - \langle r_i(s-)\rangle\big)^2 ds\right)\right.$$

$$\left. + \sum_{i=p+1}^{n}\left(\int_0^t \ln\left(\frac{\theta(i|\alpha)}{\langle\theta_j(s-)\rangle}\right) d\tilde{N}_i(s) - \int_0^t \big(\theta(i|\alpha) - \langle\theta_j(s-)\rangle\big)\,ds\right)\right]. \quad (4)$$

Large time behavior

Assumption (ND): For any (α, β) with $\alpha \neq \beta$ there exists $i \in \{0, 1, \dots, n\}$ such that

- either $r(i|\alpha) \neq r(i|\beta)$, if $i \leq p$ or $\theta(i|\alpha) \neq \theta(i|\beta)$, if $i > p$.

Theorem

Under Assumption (ND), there exist random variables $q_\alpha(\infty), \alpha \in \mathcal{H}_P$ which takes values in $\{0,1\}$ such that

$$\lim_{t\to\infty} q_\alpha(t) \quad = \quad q_\alpha(\infty), \ \forall \alpha \in \mathcal{H}_P, a.s \tag{5}$$

The random variables $q_\alpha(\infty), \alpha \in \mathcal{H}_P$, satisfy

$$\mathbb{P}[q_\alpha(\infty) = 1] \quad = \quad q_0(\alpha), \forall \alpha \in \mathcal{H}_P, \tag{6}$$

$$q_\alpha(\infty) q_\beta(\infty) \quad = \quad 0, \ \forall \alpha \neq \beta, \ a.s. \tag{7}$$

As a consequence there exists a random variable Υ with values in $\mathcal{H}_P$ such that $\mathbb{P}(\Upsilon = \alpha) = q_\alpha(0)$ and such that $\lim_{t\to\infty} \rho(t-) = |\Upsilon\rangle\langle\Upsilon|, \ a.s.$

Elements of proof

- Since $q_\alpha(t)$ is bounded and almost surely convergent we have that

$$\mathbb{E}[q_\alpha(t)^2] \quad \text{converges when t goes to infinity}$$

- This implies that

$$\begin{aligned}
\int_0^\infty \mathbb{E}\left[q_\alpha(s)^2(r(i|\alpha) - \langle r_i(s)\rangle)^2\right] ds &< \infty,\ i = 0, \dots, p, \\
\int_0^\infty \mathbb{E}\left[q_\alpha(s)^2 \left(\frac{\theta(i|\alpha)}{\langle\theta_i(s)\rangle} - 1\right)^2 \langle\theta_i(s)\rangle\right] ds &< \infty,\ i = p+1, \dots, n. \quad (8)
\end{aligned}$$

- This implies that

$$\begin{aligned}
\lim_{t\to\infty} q_\alpha(t)^2(r(i|\alpha) - \langle r_i(t)\rangle)^2 &= 0,\ i = 0 \dots, p \quad (9) \\
\lim_{t\to\infty} q_\alpha(t)^2(\theta(i|\alpha) - \langle\theta_i(t)\rangle)^2 &= 0,\ i = p+1, \dots, n. \quad (10)
\end{aligned}$$

Elements of proof

- This gives

$$q_\alpha(\infty)(r(i|\alpha) - \langle r_i(\infty)\rangle) = 0, \ i = 0\dots, p \quad (11)$$
$$q_\alpha(\infty)\,(\theta(i|\alpha) - \langle\theta_i(\infty)\rangle) = 0 \ i = p+1, \dots, n. \quad (12)$$

- This way, almost surely, for all $\alpha \neq \beta$

$$q_\alpha(\infty)\big(r(i|\alpha) - \langle r_i(\infty)\rangle\big) = q_\beta(\infty)\big(r(i|\beta) - \langle r_i(\infty)\rangle\big) = 0 \quad (13)$$
$$q_\alpha(\infty)\big(\theta(i|\alpha) - \langle\theta_i(\infty)\rangle\big) = q_\beta(\infty)\big(\theta(i|\beta) - \langle\theta_i(\infty)\rangle\big) = 0. \quad (14)$$

It follows that, almost surely, for all $\alpha \neq \beta$

$$q_\alpha(\infty)q_\beta(\infty)(r(i|\alpha) - r(i|\beta)) = 0, \ i = 0\dots, p \quad (15)$$
$$q_\alpha(\infty)q_\beta(\infty)(\theta(i|\alpha) - \theta(i|\beta)) = 0, \ i = p+1, \dots, n. \quad (16)$$

- Since it was assumed that for all $\alpha \neq \beta$ there exist i such that $r(i|\alpha) \neq r(i|\beta)$ or $\theta(i|\alpha) \neq \theta(i|\beta)$ we get $q_\alpha(\infty)q_\beta(\infty) = 0$

Elements of proof

- Let $\omega \in \Omega$. Only one $q_\alpha(\infty)(\omega)$ can be non zero and since $\sum_\alpha q_\alpha(\infty) = 1$ we get $q_\alpha(\infty)(\omega) = 1$ and $q_\beta(\omega) = 0$. We define then

$$\Upsilon(\omega) = \alpha$$

- We have $P(\Upsilon = \alpha) = P(q_\alpha(\infty) = 1) = \mathbb{E}[q_\alpha(\infty)] = q_0(\alpha)$
- Since $q_\Upsilon(\infty) = 1$ and since $q_\Upsilon(t) = Tr(\rho(t-)|\Upsilon\rangle\langle\Upsilon|)$ we have

$$\lim_{t\to\infty} Tr(\rho(t-)|\Upsilon\rangle\langle\Upsilon|) = 1, \text{ a.s.}$$

and then we can deduce that

$$\lim_{t\to\infty} \rho(t-) = |\Upsilon\rangle\langle\Upsilon|, \text{ a.s.}$$

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and then we can deduce that

$$\lim_{t\to\infty} \rho(t-) = |\Upsilon\rangle\langle\Upsilon|, \text{ a.s.}$$

Interpretation

- At time 0 $\phi_0 = \sum_\alpha \langle \alpha, \phi_0 \rangle |\alpha\rangle$.
- A direct Von Neumann measurement along the basis $|\alpha\rangle$ say that ϕ_0 is equal $|\alpha\rangle$ with probability $q_0(\alpha)$ which is equivalent to say that $\rho_0 = |\phi_0\rangle\langle\phi_0| = |\alpha\rangle\langle\alpha|$ with probability $q_0(\alpha) = Tr(\rho_0 |\alpha\rangle\langle\alpha|)$.
- The last theorem says that $(\rho(t-))$ behaves in large time as a random state which obeys to the same rules of the state resulting of a direct measurement at time 0.

Rate of convergence

Theorem

Assume that $\theta(i|\alpha) > 0$, for all $\alpha \in \mathcal{H}_P$ and for all $i = p+1, \ldots, n$. Assume Assumption (ND) is satisfied and assume that $q_\alpha(0) \neq 0$, for all $\alpha \in \mathcal{H}_P$. Let $\gamma \in \mathcal{H}_P$, we have

$$\lim_{t\to\infty} \frac{1}{t} \ln\left(\frac{q_\alpha(t)}{q_\Upsilon(t)}\right) = -\frac{1}{2}\sum_{i=0}^{p}(r(i|\alpha) - r(i|\Upsilon))^2 + \sum_{p+1}^{n} \theta(i|\Upsilon)\left[1 - \frac{\theta(i|\alpha)}{\theta(i|\Upsilon)} + \ln\left(\frac{\theta(i|\alpha)}{\theta(i|\Upsilon)}\right)\right], \quad (17)$$

for all $\alpha \in \mathcal{H}_P$, $\mathbb{P}$ almost surely.

Elements of proof

- $$q_\alpha(t) = q_0(\alpha) \times \exp\Bigg[\sum_{i=0}^{p}\Bigg(\int_0^t \big(r(i|\alpha) - \langle r_i(s-)\rangle\big)\, dW_i(s) - \frac{1}{2}\int_0^t \big(r(i|\alpha) - \langle r_i(s-)\rangle\big)^2 ds\Bigg) + \sum_{i=p+1}^{n}\Bigg(\int_0^t \ln\left(\frac{\theta(i|\alpha)}{\langle\theta_j(s-)\rangle}\right) d\tilde{N}_i(s) - \int_0^t \big(\theta(i|\alpha) - \langle\theta_j(s-)\rangle\big)\, ds\Bigg)\Bigg]. \quad (18)$$

- $$\frac{q_\alpha(t)}{q_\gamma(t)} = \frac{q_\alpha(0)}{q_\gamma(0)} \times \exp\Bigg[\sum_{i=0}^{p}\Bigg(\int_0^t (r(i|\alpha) - r(i|\gamma)) dW_i(s) - \frac{1}{2}\int_0^t \Big[(r(i|\alpha) - \langle r_i(s-)\rangle)^2 - (r(i|\gamma) - \langle r_i(s-)\rangle)^2\Big] ds\Bigg) + \sum_{i=p+1}^{n}\Bigg(\int_0^t \ln\left(\frac{\theta(i|\alpha)}{\theta(i|\gamma)}\right) dN_i(s) - \int_0^t (\theta(i|\alpha) - \theta(i|\gamma))\, ds\Bigg)\Bigg]. \quad (19)$$

Elements of proof

This can be rewritten as

$$\frac{q_\alpha(t)}{q_\gamma(t)} = \frac{q_\alpha(0)}{q_\gamma(0)} \times \exp\Bigg[\sum_{i=0}^{p}\Bigg(\int_0^t (r(i|\alpha) - r(i|\gamma))dX_i^\gamma(s) - \frac{1}{2}\int_0^t (r(i|\alpha) - r(i|\gamma))^2 ds\Bigg) + \sum_{i=p+1}^{n}\Bigg(\int_0^t \ln\left(\frac{\theta(i|\alpha)}{\theta(i|\gamma)}\right) d\hat{M}_i^\gamma(s) - \int_0^t (\theta(i|\alpha) - \theta(i|\gamma)) - \ln\left(\frac{\theta(i|\alpha)}{\theta(i|\gamma)}\right)\theta(i|\gamma))ds\Bigg)\Bigg], \tag{20}$$

for all $t \geqslant 0$, where

$$\begin{aligned} X_i^\gamma(t) &= W_i(t) - \int_0^t \Big[r(i|\gamma) - \langle r_i(s-)\rangle\Big] ds, \; i = 0, \dots, p \\ \hat{M}_i^\gamma(t) &= N_i(t) - \int_0^t \theta(i|\gamma)ds = N_i(t) - \theta(i|\gamma)t, \; i = p+1, \dots, n \end{aligned} \tag{21}$$

Elements of proof

- Change of measure

$$d\mathbb{Q}_\gamma^t(\omega)=\frac{q_\gamma(t)(\omega)}{q_\gamma(0)}d\mathbb{P}(\omega)\ t\geqslant 0,$$

- This family of probability measure is consistent. If $s\leqslant t$, $\mathbb{E}_{\mathbb{Q}_\gamma^t}[A]=\mathbb{E}_{\mathbb{Q}_\gamma^s}[A],\forall A\in\mathcal{F}_s$.
- Moreover, since $\mathbb{E}[q_\gamma(\infty)|\mathcal{F}_t]=q_\gamma(t)$, for all $t\geqslant 0$, any element of this family can be extended to a unique probability measure $\mathbb{Q}_\gamma$ on $\mathcal{F}_\infty=\mathcal{F}$. In particular, we have

$$d\mathbb{Q}_\gamma(\omega)=\frac{q_\gamma(\infty)(\omega)}{q_\gamma(0)}d\mathbb{P}(\omega)\tag{22}$$

and in terms of filtration we get the following Radon Nykodim formula

$$\mathbb{E}\left[\frac{d\mathbb{Q}_\gamma(\omega)}{d\mathbb{P}(\omega)}\middle|\mathcal{F}_t\right]=\frac{q_\gamma(t)}{q_\gamma(0)}=\frac{d\mathbb{Q}_\gamma^t(\omega)}{d\mathbb{P}(\omega)},\ t\geqslant 0.\tag{23}$$

Elements of proof

In the sequel we want to study $\lim_t \frac{1}{t} \ln\left(\frac{q_\alpha(t)}{q_\gamma(t)}\right)$ under $\mathbb{Q}_\gamma$

$$\begin{aligned} X_i^\gamma(t) &= W_i(t) - \int_0^t \Big[r(i|\gamma) - \langle r_i(s-)\rangle\Big] ds, \ \ i = 0,\dots,p \\ \hat{M}_i^\gamma(t) &= N_i(t) - \int_0^t \theta(i|\gamma) ds = N_i(t) - \theta(i|\gamma)t, \ \ i = p+1,\dots,n \end{aligned} \tag{24}$$

Lemme

Let $\gamma \in \mathcal{P}$ such that $q_\gamma(0) \neq 0$. Under $\mathbb{Q}_\gamma$, the processes $(X_j^\gamma(t)), j = 0,\dots,p$ and $(\hat{M}_j^\gamma(t)), j = p+1,\dots,n$ are $(\mathcal{F}_t)$ martingales.
More precisely $(X_j^\gamma(t)), j = 0,\dots,p$ are standard $\mathbb{Q}_\gamma$ Brownian motions and $(N_j(t)), j = p+1,\dots,n$ are usual Poisson processes with deterministic intensities $\theta(i|\gamma)$.

Elements of proof

- Taking the ln and dividing by t we get

$$\frac{1}{t}\ln\left(\frac{q_\alpha(t)}{q_\gamma(t)}\right) = \frac{1}{t}\ln\left(\frac{q_\alpha(0)}{q_\gamma(0)}\right) + \sum_{i=0}^{p}\left[(r(i|\alpha)-r(i|\gamma))\frac{X_j^\gamma(t)}{t} - \frac{1}{2}(r(i|\alpha)-r(i|\gamma))^2\right] + \sum_{i=p+1}^{n}\ln\left(\frac{\theta(i|\alpha)}{\theta(i|\gamma)}\right)\frac{\hat{M}_i^\gamma(t)}{t} + (\theta(i|\gamma)-\theta(i|\alpha)) + \ln\left(\frac{\theta(i|\alpha)}{\theta(i|\gamma)}\right)\theta(i|\gamma)$$

- This way, using law of large number for Brownian motion and Poisson processes, we get $\mathbb{Q}_\gamma$ almost surely

$$\lim_{t\to\infty}\frac{1}{t}\ln\left(\frac{q_\alpha(t)}{q_\gamma(t)}\right) = -\frac{1}{2}\sum_{i=0}^{p}(r(i|\alpha)-r(i|\gamma))^2 + \sum_{i=p+1}^{n}\theta(i|\gamma)\left[1-\frac{\theta(i|\alpha)}{\theta(i|\gamma)}+\ln\left(\frac{\theta(i|\alpha)}{\theta(i|\gamma)}\right)\right], \quad (25)$$

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$$\frac{1}{t}\ln\left(\frac{q_\alpha(t)}{q_\gamma(t)}\right)=\frac{1}{t}\ln\left(\frac{q_\alpha(0)}{q_\gamma(0)}\right)$$
$$+\sum_{i=0}^{p}\left[(r(i|\alpha)-r(i|\gamma))\frac{X_j^\gamma(t)}{t}-\frac{1}{2}(r(i|\alpha)-r(i|\gamma))^2\right]$$
$$+\sum_{i=p+1}^{n}\ln\left(\frac{\theta(i|\alpha)}{\theta(i|\gamma)}\right)\frac{\hat{M}_i^\gamma(t)}{t}+(\theta(i|\gamma)-\theta(i|\alpha))+\ln\left(\frac{\theta(i|\alpha)}{\theta(i|\gamma)}\right)\theta(i|\gamma)$$

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Elements of proof

$$d\mathbb{P}(\omega) = \sum_{\gamma \ s.t. \ q_\gamma(0)>0} q_\gamma(0) d\mathbb{P}(\omega | \Upsilon(\omega) = \gamma)$$

which yields

$$d\mathbb{Q}_\gamma(\omega) = d\mathbb{P}(\omega | \Upsilon(\omega) = \gamma).$$

As a consequence

$$\lim_{t\to\infty} \frac{1}{t} \ln\left(\frac{q_\alpha(t)}{q_\Upsilon(t)}\right) = -\frac{1}{2}\sum_{i=0}^{p}(r(i|\alpha) - r(i|\Upsilon))^2 + \sum_{i=p+1}^{n} \theta(i|\Upsilon)\left[1 - \frac{\theta(i|\alpha)}{\theta(i|\Upsilon)} + \ln\left(\frac{\theta(i|\alpha)}{\theta(i|\Upsilon)}\right)\right], \quad (26)$$

$\mathbb{P}$ almost surely

$$q_\alpha(t) = e^{-t\left[\frac{1}{2}\sum_{i=0}^{p}(r(i|\alpha)-r(i|\Upsilon))^2 - \sum_{i=p+1}^{n}\theta(i|\Upsilon)\left[1-\frac{\theta(i|\alpha)}{\theta(i|\Upsilon)}+\ln\left(\frac{\theta(i|\alpha)}{\theta(i|\Upsilon)}\right)\right]+\circ(1)\right]} \times (1+\circ(1)), \quad (27)$$

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Can jumps beat the diffusion?

- $$d\rho(t-) = L(\rho(t-))dt + \Big(C\rho(t-) + \rho(t-)C^* - \mathrm{Tr}[(C+C^*)\rho(t-)]\rho(t-)\Big)dW_t \quad (28)$$

  or

  $$d\rho(t-) = L(\rho(t-))dt + \left(\frac{C\rho(t-)C^*}{\mathrm{Tr}[C^*C\rho(t-)]} - \rho(t-)\right)[d\hat{N}(t) - \mathrm{Tr}[C^*C\rho(t-)]dt]. \quad (29)$$

- A simple study shows that

  $$(c(\alpha) - c(\Upsilon))^2 \leqslant -c(\Upsilon)^2[\ln(c(\alpha)^2/c(\Upsilon)^2) + 1 - c(\alpha)^2/c(\Upsilon)^2].$$

- But it comes at a price. Suppose C has two different eigenvalues of equal norm: $c(\alpha) \neq c(\beta)$, $|c(\alpha)| = |c(\beta)|$. The non degeneracy assumption (ND) is not fulfilled for the jump equation (29) whereas it is fulfilled for the diffusive equation (28)

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Problem of Estimation

Estimation

- What happens if we do not know the initial state $\rho_0(t)$. We cannot describe the true trajectory ($\rho(t-)$) but we can describe an estimation ($\tilde{\rho}(t-)$).

$$\begin{aligned} d\rho(t-) &= L(\rho(t-))dt + \sum_{i=0}^{p} H_i(\rho(t-))(dy_i(t) - \mathrm{Tr}[(C_i + C_i^*)\rho(t-)]dt) \\ &+ \sum_{i=p+1}^{n} \left(\frac{J_i(\rho(t-))}{v_i(\rho(t-))} - \rho(t-) \right) (dN_i(t) - v_i(\rho(t-))dt). \\ \rho(0) &= \rho_0 \end{aligned}$$

$$\begin{aligned} d\tilde{\rho}(t-) &= L(\tilde{\rho}(t-))dt + \sum_{i=0}^{p} H_i(\tilde{\rho}(t-))(dy_i(t) - \mathrm{Tr}[(C_i + C_i^*)\tilde{\rho}(t-)]dt) \\ &+ \sum_{i=p+1}^{n} \left(\frac{J_i(\tilde{\rho}(t-))}{v_i(\tilde{\rho}(t-))} - \tilde{\rho}(t-) \right) (dN_i(t) - v_i(\tilde{\rho}(t-))dt). \\ \tilde{\rho}(0) &= \tilde{\rho}_0, \end{aligned}$$

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\begin{aligned}
d\rho(t-) &= L(\rho(t-))dt + \sum_{i=0}^{p} H_i(\rho(t-))(dy_i(t) - \mathrm{Tr}[(C_i + C_i^*)\rho(t-)]dt) \\
&+ \sum_{i=p+1}^{n} \left(\frac{J_i(\rho(t-))}{v_i(\rho(t-))} - \rho(t-) \right) (dN_i(t) - v_i(\rho(t-))dt). \\
\rho(0) &= \rho_0
\end{aligned}
$$

$$
\begin{aligned}
d\tilde{\rho}(t-) &= L(\tilde{\rho}(t-))dt + \sum_{i=0}^{p} H_i(\tilde{\rho}(t-))(dy_i(t) - \mathrm{Tr}[(C_i + C_i^*)\tilde{\rho}(t-)]dt) \\
&+ \sum_{i=p+1}^{n} \left(\frac{J_i(\tilde{\rho}(t-))}{v_i(\tilde{\rho}(t-))} - \tilde{\rho}(t-) \right) (dN_i(t) - v_i(\tilde{\rho}(t-))dt). \\
\tilde{\rho}(0) &= \tilde{\rho}_0,
\end{aligned}
$$

- Define $\tilde{q}_\alpha(t) = Tr(\tilde{\rho}(t-)|\alpha\rangle\alpha|)$

$$d\tilde{q}_\alpha(t) = \tilde{q}_\alpha(t-)\Bigg[\sum_{i=0}^{p}\Big(r(i|\alpha)-\langle\tilde{r}_i(t-)\rangle\Big)\Big(dW_i(t)+(\langle r_i(t-)\rangle-\langle\tilde{r}_i(t-)\rangle)\Big) + \sum_{i=p+1}^{n}\left(\frac{\theta(i|\alpha)}{\langle\tilde{\theta}_i(t-)\rangle}-1\right)\Big(dN_i(t)-\langle\tilde{\theta}_i(t-)\rangle dt\Big)\Bigg]. \quad (30)$$

- They are no more martingales

$$\tilde{q}_\alpha(t) = \tilde{q}_0(\alpha)\times\exp\Bigg[\sum_{i=0}^{p}\Bigg(\int_0^t\Big(r(i|\alpha)-\langle\tilde{r}_i(s-)\rangle\Big)\Big(dW_i(s)+(\langle r_i(s-)\rangle-\langle\tilde{r}_i(s-)\rangle)ds - \frac{1}{2}\int_0^t\Big(r(i|\alpha)-\langle\tilde{r}_i(s-)\rangle\Big)^2 ds\Bigg) + \sum_{i=p+1}^{n}\Bigg(\int_0^t \ln\left(\frac{\theta(i|\alpha)}{\langle\tilde{\theta}_i(s-)\rangle}\right)dN_i(s) - \int_0^t\Big(\theta(i|\alpha)-\langle\tilde{\theta}_i(s-)\rangle\Big))ds\Bigg)\Bigg], \quad (31)$$

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$$
\begin{aligned}
&\frac{\tilde{q}_\alpha(t)}{\tilde{q}_\gamma(t)} = \\
&\frac{\tilde{q}_\alpha(0)}{\tilde{q}_\gamma(0)} \times \exp\Bigg[\sum_{i=0}^{p}\left(\int_0^t \big(r(i|\alpha) - r(i|\gamma)\big) dX_i^\gamma(s) - \frac{1}{2}\int_0^t (r(i|\alpha) - r(i|\gamma))^2 ds\right) \\
&+ \sum_{i=p+1}^{n}\left(\int_0^t \ln\left(\frac{\theta(i|\alpha)}{\theta(i|\gamma)}\right) d\hat{M}_i^\gamma(s) - \int_0^t (\theta(i|\alpha) - \theta(i|\gamma)) - \ln\left(\frac{\theta(i|\alpha)}{\theta(i|\gamma)}\right)\theta(i|\gamma))ds\right)\Bigg],
\end{aligned}
\tag{32}
$$

Proposition

Assume that $\theta(i|\alpha) > 0$, *for all* $\alpha \in \mathcal{H}_P$ *and for all* $i = p+1, \dots, n$. *Assume Assumption (ND) is satisfied and assume that* $\tilde{q}_\alpha(0) \neq 0$, *for all* $\alpha \in \mathcal{H}_P$. *Let* $(\tilde{q}_\alpha(t))$ *be the stochastic processes defined by* (31). *We have*

$$\lim_{t\to\infty} \tilde{q}_\Upsilon(t) \quad = \quad 1, \lim_{t\to\infty} \tilde{q}_\alpha(t)\mathbf{1}_{\Upsilon\neq\alpha} = 0, \tag{33}$$

$$\lim_{t\to\infty} \frac{1}{t} \ln\left(\frac{\tilde{q}_\alpha(t)}{\tilde{q}_\Upsilon(t)}\right) = -\frac{1}{2}\sum_{i=0}^{p}(r(i|\alpha) - r(i|\Upsilon))^2 + \sum_{p+1}^{n} \theta(i|\Upsilon)\left[1 - \frac{\theta(i|\alpha)}{\theta(i|\Upsilon)} + \ln\left(\frac{\theta(i|\alpha)}{\theta(i|\Upsilon)}\right)\right], \tag{34}$$

$\mathbb{P}$ *almost surely.*
Finally $\lim_{t\to\infty} \tilde{\rho}(t) = |\Upsilon\rangle\langle\Upsilon|, \mathbb{P}$ *almost surely.*

III) From discrete to continuous model

Setup

- Can we recover the continuous time model from the discrete one (yes we already know it)
- Under some renormalization, when τ goes to zero the discrete time model converges to the continuous time model
- Remember that the interaction bas been described by

$$U = \sum_{\alpha} |\alpha\rangle\langle\alpha| \otimes U_\alpha$$

- Put $\tau = h$ and consider an hamiltonian of the form

$$H = \frac{1}{\sqrt{h}} \sum_{\alpha} |\alpha\rangle\langle\alpha| \otimes H_\alpha, \text{ and} \quad (35)$$

$$U = e^{-ihH} \quad (36)$$

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- Recall that with probability $\pi_n(j) = \sum_\beta q_n(\beta)p(j|\beta)$

$$q_{n+1}(\alpha)(j) = \frac{q_n(\alpha)p(j|\alpha)}{\pi_n(j)} = \frac{q_n(\alpha)p(j|\alpha)}{\sum_\beta q_n(\beta)p(j|\beta)},$$

with $p(j|\alpha) = |\langle j, U_\alpha \psi\rangle|^2$

- Developing $p(j|\alpha)$ in terms of h, we get

$$\begin{aligned} p(j|\alpha) &= |\langle j|U_\alpha|\psi\rangle|^2 \\ &= |\langle j|I - i\sqrt{h}H_\alpha - \frac{1}{2}h(H_\alpha)^2|\psi\rangle|^2 \\ &= |\langle j|\psi\rangle|^2 + i\sqrt{h}\left(\langle j|\psi\rangle\overline{\langle j|H_\alpha\psi\rangle} - \overline{\langle j|\psi\rangle}\langle j|H_\alpha\psi\rangle\right) \\ &\quad + h\left(-\frac{1}{2}\langle j|\psi\rangle\overline{\langle j|H_\alpha^2\psi\rangle} - \frac{1}{2}\overline{\langle j|\psi\rangle}\langle j|H_\alpha^2\psi\rangle + |\langle j|H_\alpha\psi\rangle|^2\right) \end{aligned}$$

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$$q_{n+1}(\alpha)(j) = \frac{q_n(\alpha)p(j|\alpha)}{\pi_n(j)} = \frac{q_n(\alpha)p(j|\alpha)}{\sum_\beta q_n(\beta)p(j|\beta)},$$

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- If $\langle j, \psi \rangle = 0$

$$p(j|\alpha) = h|\langle j|H_\alpha \psi\rangle|^2 = h\Omega(j,\alpha)$$

- Otherwise

$$p(j|\alpha) = |\langle j\psi\rangle|^2 \left[1 - \sqrt{h}\Gamma(j|\alpha) + h[-\Delta(j|\alpha) + \Theta(j,\alpha)\right]$$

with

$$\Gamma(j,\alpha) = 2Im\left(\frac{\langle j|H_\alpha\psi\rangle}{\langle j|\psi\rangle}\right) \quad (37)$$

$$\Delta(j,\alpha) = Re\left(\frac{\langle j|H_\alpha^2\psi\rangle}{\langle j|\psi\rangle}\right) \quad (38)$$

$$\Theta(j,\alpha) = \left|\frac{\langle j|H_\alpha\psi\rangle}{\langle j|\psi\rangle}\right|^2 \quad (39)$$

- Consider the case $i = 0, 1$ and h small
- In the case $\psi = |0\rangle$ we will often observe $|0\rangle$ and sometimes $|1\rangle$ which is typical to a jump evolution
- If $\langle 0, \psi\rangle$ and $\langle 1, \psi\rangle$ are non zero the probability of observing $|0\rangle$ or $|1\rangle$ are comparable and the terms $\sqrt{h}$ is typical of a diffusive evolution.

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- The sequence $q_{[t/h]}(\alpha)$ converges when h goes to zero to

$$
\begin{aligned}
dq_\alpha(t) &= q_\alpha(t-)\Bigg[\sum_i \left(\frac{\Omega(j,\alpha)}{\langle\Omega_j(t-)\rangle} - 1\right)(d\tilde{N}_t(i) - \langle\Omega_j(t-)\rangle dt) \\
&\quad + \sum_j (\Gamma(j,\alpha) - \langle\Gamma_j(t-)\rangle) dW_t(j)\Bigg],
\end{aligned}
$$

  where $\langle\Omega_j(t)\rangle = \sum_\beta q_t(\beta)\Omega(j,\beta)$ and $\langle\Gamma_j(t)\rangle = \sum_\beta q_t(\beta)\Gamma(j,\beta)$
- The convergence is in distribution for stochastic processes
- To show this result we use the convergence of Markov generators

Stochastic Master equation

In terms of Stochastic master equations, we have

$$\begin{aligned} d\rho(t-) &= L(\rho(t-))dt + \sum_{i=0}^{p} H_i(\rho(t-))dW_i(t) \\ &\quad + \sum_{i=p+1}^{n} \left(\frac{J_i(\rho(t-))}{v_i(\rho(t-))} - \rho(t-) \right) (dN_i(t) - v_i(\rho(t-))dt), \end{aligned}$$

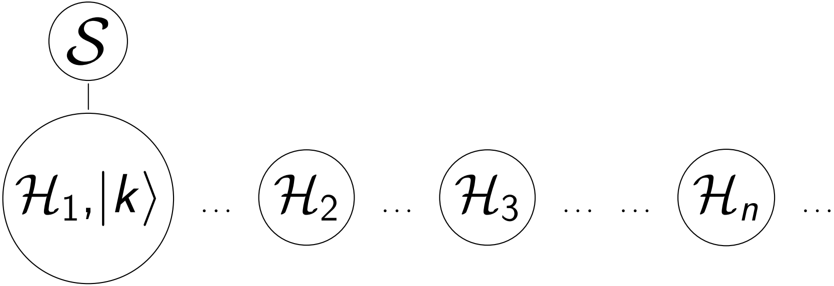
where the operator $C_j = i \sum_\alpha \langle j, H_\alpha \psi \rangle |\alpha\rangle\langle\alpha|$

A non Markovian extension

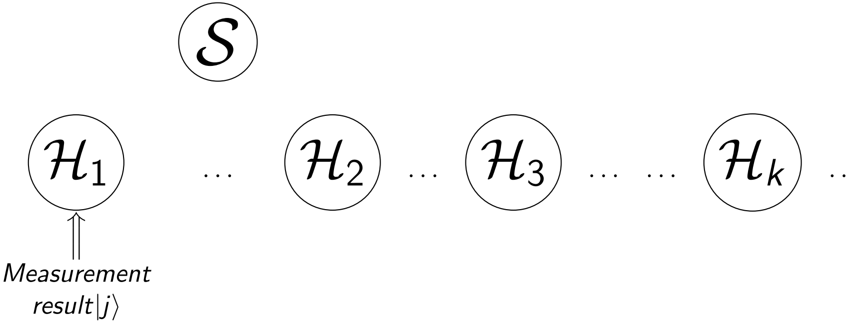
A non Markovian extension

- The following idea is just an exploration idea.
- The idea is the following. Assume that at time n the state of the piece of the chain $\mathcal{H}$ is $\beta = |k\rangle\langle k|$ for some k in the Ancilla basis.
- The state ρ_n of $\mathcal{S}$ interacts with $\mathcal{H}, \beta$: $U(\rho_n \otimes \beta)U^\star$
- If we have observed the state j during the indirect measurement, the state of $\mathcal{S} \otimes \mathcal{H}$ becomes
$$\frac{\mathcal{L}_j(\rho_n)}{\mathrm{Tr}[\mathcal{L}_j(\rho_n)]} \otimes |j\rangle\langle j|$$
- We then assume that the state of the next Ancilla is $|j\rangle\langle j|$.
- This introduces memory...

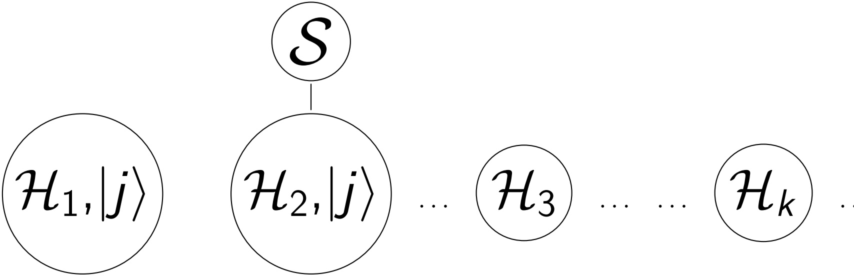
1st interaction



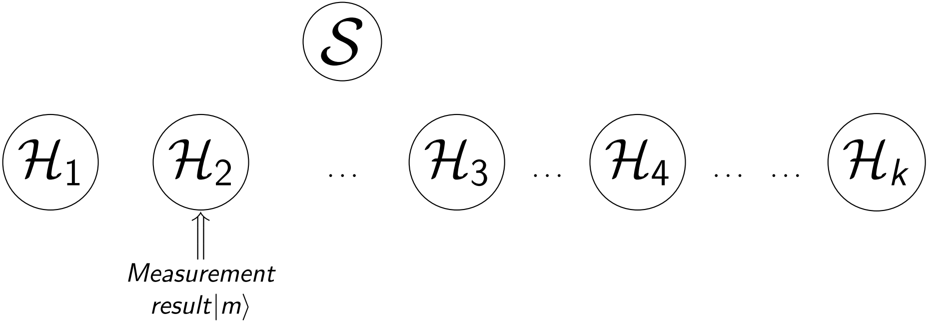
1st measurement



2nd interaction



2nd measurement



A non Markovian extension

- Assume that at time n the state of the piece of the chain $\mathcal{H}$ is $\beta = |k\rangle\langle k|$ for some k in the Ancilla basis.
- Here we introduce $p(j|k,\alpha) = |\langle j, U_\alpha k\rangle|^2$
- We have

$$\begin{aligned} p(j|k,\alpha) &= h|\langle j|H_\alpha k\rangle|^2 = h\Omega(j,\alpha,k) \quad (40)\\ p(k|k,\alpha) &= 1 - h\sum_{j\neq k}\Omega(j,\alpha,k) \quad (41)\end{aligned}$$

- Then the probability of transition are given by

$$\begin{aligned} \pi_n(j,k) &= = h\sum_{\alpha} q_n(\alpha)\Omega(j,\alpha,k) \quad (42)\\ \pi_n(k,k) &= 1 - h\sum_{j\neq k}\sum_{\alpha} q_n(\alpha)\Omega(j,\alpha,k) \quad (43)\end{aligned}$$

- Often the next state of the Ancilla does not change and sometimes there is a jump to another state

A non Markovian extension

- In order to describe the limit equation, we need also to describe the evolution of the state of the Ancillas.
- We get a couple of equations

$$\bullet\, d\rho(t-) = \sum_j -\frac{1}{2}\{(C_j(\beta_{t-}))^\star C_j(\beta_{t-}), \rho(t-)\} + C_j(\beta_{t-})\rho(t-)C_j(\beta_{t-})^\star$$

$$\sum_j \left(\frac{C_j(\beta_{t-})\rho(t-)C_j(\beta_{t-})^\star}{\mathrm{Tr}[C_j(\beta_{t-})\rho(t-)C_j(\beta_{t-})^\star]} - \rho(t-) \right) (d\tilde{N}_j(t) - \mathrm{Tr}[C_j(\beta_{t-})\rho(t-)C_j(\beta_{t-})^\star]dt)$$

$$\bullet\, d\beta_{t-} = \sum_j (|j\rangle\langle j| - \beta_{t-}))\, d\tilde{N}_j(t),$$

where

$$C_j(\beta_{t-}) = i\sum_\alpha \langle j, H_\alpha \beta_{t-}\rangle |\alpha\rangle\langle\alpha|.$$

Thank You