Algebraic Structure of Nonequilibrium Quantum Statistical Mechanics

Algebraic Structures in Nonequilibrium Quantum Statistical Mechanics

Dirac, Heisenberg, Jordan

von Neumann, Weyl

Araki-Woods '63, Araki '63, Araki-Wyss '64

Haag-Hugenholtz-Winnink '67

Tomita '67 – Takesaki '70

Connes '73

- - - - - -

[Bratteli-Robinson]

I. The Standard Representation

Finite quantum system

$\mathcal{K}$ = finite dimensional Hilbert space

$\mathcal{O} = \mathcal{B}(\mathcal{K})$ = linear operators on $\mathcal{K}$

$\mathbb{C}$-algebra

*-algebra

Banach algebra

Involution *

Norm $\|\cdot\|$

$\|A^*A\| = \|A\|^2$

C^*-algebra

$\mathcal{A} \subset \mathcal{O}$: $\quad \mathcal{A}^* = \{A^* \mid A \in \mathcal{A}\}$

$\mathcal{A}' = \{B \in \mathcal{O} \mid [A, B] = 0, \forall A \in \mathcal{A}\}$

$\mathcal{A} = \mathcal{A}^* \Rightarrow \mathcal{A}'' =$ smallest C^*-algebra $\supset \mathcal{A}$

$\mathcal{A} \vee \mathcal{B} =$ smallest C^*-algebra $\supset \mathcal{A} \cup \mathcal{B}$

$\boxed{(A \mid B) = \operatorname{tr}(A^* B)}$ makes $\mathcal{O}$ a Hilbert space

<u>Notation</u> $\mathcal{H}_{\mathcal{O}} = \mathcal{O} \simeq \mathcal{K} \otimes \mathcal{K}$

$\xi, \eta, \ldots \in \mathcal{H}_{\mathcal{O}}$ (vectors)

$A, B, \ldots \in \mathcal{O}$ (operators)

$$L(A)\xi = A\xi \qquad R(A)\xi = \xi A^*$$

make $\mathcal{H}_{\mathcal{O}}$ an injective $\mathcal{O}$-module, i.e.,

$L : \mathcal{O} \to \mathcal{L}(\mathcal{H}_{\mathcal{O}})$ faithful representation

$$(\xi | L(A^*)\eta) = (L(A)\xi | \eta) = (\xi | L(A)^* \eta)$$

Standard Representation

Proposition 4.

a) L is isometric, $\|L(A)\| = \|A\|$

b) $L(\mathcal{O})$ & $R(\mathcal{O})$ are isomorphic to $\mathcal{O}$

c) $L(\mathcal{O}) \cap R(\mathcal{O}) = \mathbb{C}\, I$

d) $L(\mathcal{O}) \vee R(\mathcal{O}) = \mathcal{L}(\mathcal{H}_0)$

e) $L(\mathcal{O})' = R(\mathcal{O})$ & $R(\mathcal{O})' = L(\mathcal{O})$

II. The Natural Cone.

State on $\mathcal{O}$ = linear functional

$$\varrho : \mathcal{O} \to \mathbb{C}$$

s.t. $\varrho(A^* A) \geq 0$ and $\varrho(I) = 1$

Riesz $\Rightarrow$ $\varrho(A) = (\rho \,|\, A) = \operatorname{tr}(\rho \, A)$

for some $\rho \in \mathcal{H}_{\mathcal{O}}$ s.t. $\rho \geq 0$ & $\operatorname{tr}(\rho) = 1$

$$\xi_\rho = \rho^{1/2} \Rightarrow \boxed{\varrho(A) = (\xi_\rho | L(A)\, \xi_\rho)}$$

ξ_ρ unit vector in $\mathcal{H}_{\mathcal{O}}^+ = \{ A \in \mathcal{O} \,|\, A \geq 0 \}$

$\mathcal{H}_{\mathcal{O}}^+$ is the Natural Cone

$$C = \mathbb{R}_+ C \subset \mathcal{H} \rightsquigarrow \begin{cases} \hat{C} = \{ \xi \in \mathcal{H} \mid (\eta | \xi) \geq 0 \ \forall \eta \in C \} \\ \text{Dual cone} \end{cases}$$

$$J : \mathcal{H}_\theta \to \mathcal{H}_\theta \quad \text{(anti-unitary involution)}$$
$$\xi \mapsto \xi^*$$

is the Modular Conjugation

<u>Proposition 2.</u>

a) $\rho \mapsto \xi_\rho$ is a bijection, $\|\xi_\rho - \xi_\nu\|^2 \leq \|\rho - \nu\|_{tr} \leq \|\xi_\rho + \xi_\nu\| \cdot \|\xi_\rho - \xi_\nu\|$

b) $J L(\theta) J^* = L(\theta)'$

c) $\widehat{\mathcal{H}_\theta^+} = \mathcal{H}_\theta^+$ self-dual cone

d) $J\xi = \xi$ for all $\xi \in \mathcal{H}_\theta^+$

e) $L(A) J L(A) \mathcal{H}_\theta^+ \subset \mathcal{H}_\theta^+$ for all $A \in \theta$

III. Standard Liouvillians.

Dynamics = $\left\{ \begin{array}{l} \text{Continuous 1-parameter} \\ \text{subgroup of } \mathrm{Aut}(\mathcal{O}) \end{array} \right\}$ $\quad t \mapsto \tau^t = e^{t\delta}$

Generator δ is a $*$-derivation:

$$\delta(AB) = \delta(A)B + A\delta(B)$$

$$\delta(A^*) = \delta(A)^* \qquad \delta(I) = 0$$

A dynamics is Inner: $\exists\, H = H^* \in \mathcal{O}$ s.t.

$$\tau^t(A) = e^{itH} A\, e^{-itH}$$

H (unique mod $\mathbb{C}I$) is the Hamiltonian and

$$\delta = i[H, \cdot\,]$$

III. Standard Liouvillians.

$$\text{Dynamics} = \left\{ \begin{array}{l} \text{Continuous 1-parameter} \\ \text{subgroup of } \mathrm{Aut}(\mathcal{O}) \end{array} \right\} \qquad t \mapsto \tau^t = e^{t\delta}$$

Generator δ is a $*$-derivation:

$$\delta(AB) = \delta(A)B + A\delta(B)$$

$$\delta(A^*) = \delta(A)^* \qquad \delta(I) = 0$$

special to finite dim. $\mathcal{O}$!

A dynamics is inner: $\exists H = H^* \in \mathcal{O}$ s.t.

$$\tau^t(A) = e^{itH} A e^{-itH}$$

H (unique mod $\mathbb{C}I$) is the Hamiltonian and

$$\delta = i[H, \cdot]$$

$\xi \in \mathcal{H}_\sigma^+$ is cyclic : $L(\mathcal{O})\xi = \mathcal{H}_\sigma$

$$\Updownarrow$$

$\xi > 0$

$$\Updownarrow$$

ξ is separating : $L(A)\xi = 0 \Rightarrow A = 0$

$$A_t = \tau^t(A) \qquad \rho_t = e^{-itH} \rho \, e^{itH}$$

$$\rho_t(A) = \rho(A_t)$$

$$\xi_{\rho_t} = \rho_t^{1/2} = e^{-itH} \xi_\rho \, e^{itH} = L(e^{-itH}) R(e^{-itH}) \, \xi_\rho$$

$$= e^{-itL(H)} e^{itR(H)} \xi_\rho$$

$$= e^{-it(L(H) - R(H))} \xi_\rho$$

$\boxed{K = L(H) - R(H)}$ is the standard Liouvillian

Characterized by:

(i) $e^{itK} L(A) e^{-itK} = L(\tau^t(A))$

(ii) $e^{itK} \mathcal{H}_\theta^+ \subset \mathcal{H}_\theta^+$

IV. The modular group

$$\rho = e^{-\beta H}/\mathrm{tr}(-) \iff \begin{cases} \rho(A\,\tau^{i\beta}(B)) = \rho(BA) \\ \beta - \mathrm{KMS} \end{cases}$$

ρ faithful state $\rightsquigarrow$ $\sigma_\rho^t(A) = e^{t\delta_\rho}(A) = \rho^{it} A \rho^{-it}$

σ_ρ is the modular group of ρ

$\delta_\rho = i[\log \rho, \cdot\,]$ the modular derivation

Characterization: σ_ρ = unique dynamics for which ρ is (-1)-KMS

Standard Liouvillian of σ_ρ :

$$K_\rho = L(\log \rho) - R(\log \rho)$$

$\Delta_\rho = e^{K_\rho}$ is the modular operator of ρ

$$\Delta_\rho(A) = \rho A \rho^{-1} \qquad L(\sigma_\rho^t(A)) = \Delta_\rho^{it} L(A) \Delta_\rho^{-it}$$

$$\Delta_\rho^{it} \mathcal{H}_\theta^+ \subset \mathcal{H}_\theta^+$$

Characterization: $\boxed{J \Delta_\rho^{1/2} L(A) \xi_\rho = L(A^*) \xi_\rho}$

V. Relative Modular Operators

$[D\rho : D\nu]^t = \rho^{it}\nu^{-it}$ is a Connes Cocycle

$\ell_{\rho|\nu} = \frac{1}{i}\frac{d}{dt}[D\rho : D\nu]^t\Big|_{t=0} = \log\rho - \log\nu$ is the Relative Hamiltonian

Properties:

a) $([D\rho : D\nu]^t)^{-1} = [D\nu : D\rho]^t$

b) $[D\rho : D\nu]^t [D\nu : D\omega]^t = [D\rho : D\omega]^t$

c) $[D\rho : D\nu]^t \sigma_\nu^t([D\rho : D\nu]^s) = [D\rho : D\nu]^{t+s}$

d) $[D\rho : D\nu]^t \sigma_\nu^t(A)[D\nu : D\rho]^t = \sigma_\rho^t(A)$

e) $\nu(A[D\rho : D\nu]^{-i}) = \rho(A)$

Relative modular dynamics:

$$\sigma^t_{\rho|\nu}(A) = \rho^{it} A \nu^{-it} = [D\rho : D\nu]^t \sigma^t_\nu(A)$$
$$= \sigma_\rho^t(A) [D\rho : D_\nu]^t$$

has standard Liouvillian

$$K_{\rho|\nu} = L(\log \rho) - R(\log \nu)$$

The relative modular operator is $\Delta_{\rho|\nu} = e^{K_{\rho|\nu}}$

and is characterized by: $\boxed{J\Delta^{1/2}_{\rho|\nu} L(A)\xi_\nu = L(A^*)\xi_\rho}$

Note: $\log \Delta_{\rho|\nu} = \log \Delta_\nu + L(\ell_{\rho|\nu})$

$\log \Delta_\rho = \log \Delta_\nu + L(\ell_{\rho|\nu}) - R(\ell_{\rho|\nu})$

VI. Non-Commutative L^p-spaces

"Flat" spaces: $L^p(\mathcal{O}) = \mathcal{O}$ with $\|\xi\|_p = \mathrm{tr}((\xi^*\xi)^{p/2})^{1/p}$

are bad in TD-limit!

Araki-Masuda: $L^p(\mathcal{O}, \omega) = \mathcal{O}$ with $\|\xi\|_{\omega,p} = \|\xi\omega^{\frac{1}{p}-\frac{1}{2}}\|_p$

$T_p : \xi \longmapsto \xi\omega^{\frac{1}{p}-\frac{1}{2}}$ maps isometrically $L^p(\mathcal{O}) \to L^p(\mathcal{O}, \omega)$

$L^2(\mathcal{O}, \omega) = L^2(\mathcal{O}) = \mathcal{H}_\mathcal{O}$ carries the standard rep.

For $p > 2$: $\|\xi\|_{\omega,p} = \max_{\nu} \|\Delta_{\nu|\omega}^{\frac{1}{2}-\frac{1}{p}}\xi\|$

For $p \in [1, 2[$: $L^p(\mathcal{O}, \omega)$ dual to $L^q(\mathcal{O}, \omega)$, $\frac{1}{p} + \frac{1}{q} = 1$

VI. Non-Commutative L^p-spaces

"Flat" spaces: $L^p(\mathcal{O}) = \mathcal{O}$ with $\|\xi\|_p = \mathrm{tr}((\xi^*\xi)^{p/2})^{1/p}$

are bad in TD-limit!

Araki-Masuda: $L^p(\mathcal{O},\omega) = \mathcal{O}$ with $\|\xi\|_{\omega,p} = \|\xi\omega^{\frac{1}{p}-\frac{1}{2}}\|_p$

Survives TD-limit!

For $p>2$: $\|\xi\|_{\omega,p} = \max\limits_{\nu} \|\Delta_{\nu|\omega}^{\frac{1}{2}-\frac{1}{p}}\xi\|$

For $p\in[1,2[$: $L^p(\mathcal{O},\omega)$ dual to $L^q(\mathcal{O},\omega)$, $\frac{1}{p}+\frac{1}{q}=1$

Cones

$$\mathcal{C}^p = \{ \xi \omega^{\frac{1}{2}-\frac{1}{p}} \mid \xi \in \mathcal{H}_\theta^+ \} = T_p \mathcal{H}_\theta^+ \subset L^p(\theta, \omega)$$

$$\mathcal{C}^p = \{ \lambda \Delta_{\rho/\omega}^{1/p} \xi_\omega \mid \lambda \geq 0, \rho \}$$

$\mathcal{C}^2$ = natural cone

$\widehat{\mathcal{C}^p} = \mathcal{C}^q, \quad \frac{1}{p} + \frac{1}{q} = 1$

VII. L^p-Liouvillians

Generalize Standard Liouvillian:

$$\boxed{\text{Isometric Implementation of dynamics in } L^p\text{-spaces preserving the cone } \mathcal{C}^p}$$

$$U_p^t = T_p\, e^{itK}\, T_p^{-1} : \quad U_p^t \xi = e^{-itH}\, \xi\, \omega^{\frac{1}{p}-\frac{1}{2}}\, e^{itH}\, \omega^{\frac{1}{2}-\frac{1}{p}}$$
$$= e^{-itH}\, \xi\, e^{it\, \sigma_\omega^{i(\frac{1}{2}-\frac{1}{p})}(H)}$$

$$U_p^t = e^{itK_p}, \quad K_p = L(H) - R\left(\sigma_\omega^{i(\frac{1}{2}-\frac{1}{p})}(H)\right) \quad \text{p-Liouvillian}$$

Proposition 3.

a) U_p^t is isometric on $L^p(\mathcal{O}, \omega)$

b) $U_p^t \, \mathcal{C}^p \subset \mathcal{C}^p$

c) $U_p^t \, L(A) \, U_p^{-t} = L(\tau^t(A))$

d) U_p^t is uniquely characterized by a) – c)

Note: $U_2^t = e^{itK_2}$, $K_2 = K =$ Std. Liouvillian

VIII. Perturbation Theory

$\omega \circ \tau^t = \omega$ invariant state, $\tau^t = e^{t\delta}$

$V = V^* \in \mathcal{O} \rightsquigarrow \delta_\lambda = \delta + \lambda i [V, \cdot] \rightsquigarrow \tau_\lambda^t = e^{t\delta_\lambda}$

K = standard Liouvillian for τ^t

$K_\lambda = K + \lambda V - \lambda J V J$ is std. Liouvillian for τ_λ

$K_{\lambda,p} = K + \lambda V - \lambda J \sigma_\omega^{-i(\frac{1}{2} - \frac{1}{p})}(V) J$

IX. Entropies

Relative entropy:

$$\mathrm{Ent}(\nu|\rho) = -\mathrm{tr}(\nu(\log\nu - \log\rho))$$
$$= -\nu(\ell_{\nu|\rho})$$
$$= (\xi_\nu | \log \Delta_{\rho|\nu}\, \xi_\nu)$$

Rényi relative entropy:

$$\mathrm{Ent}_\alpha(\nu|\rho) = \log \mathrm{tr}(\nu^\alpha \rho^{1-\alpha})$$
$$= \log(\xi_\rho | \Delta^\alpha_{\nu|\rho}\, \xi_\rho)$$