

Entropic Fluctuations of Quantum Dynamical Semigroups

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**Advances in Open Quantum Systems
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- 1 Background
- 2 Thermodynamics of QDS
- 3 Entropic Fluctuations
- 4 Outlook

Background 1

Statistics of Current Fluctuations – Fluctuation Relations (FR)

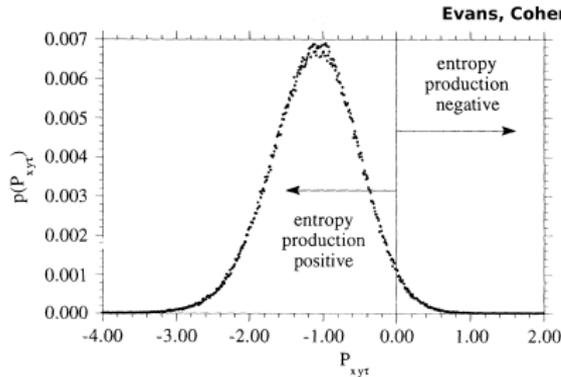


FIG. 1. The probability distribution of segment averages, $\langle P_{xy,i} \rangle_\tau$, of the xy element of the pressure tensor for 56 WCA disks at $H_0/N=1.56032$, $n=0.8$, a shear rate $\gamma=0.5$, and a segment time $\tau=0.1$. For those states where $\langle P_{xy,i} \rangle_\tau = P_{xy\tau}$ is positive the entropy production is negative for a period of time τ , counter to the second law of thermodynamics.

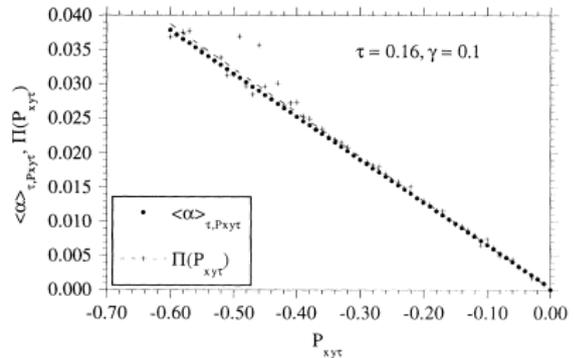


FIG. 2. The logarithmic probability ratio $\Pi(P_{xy\tau})$ and $\langle \alpha \rangle_{\tau, P_{xy\tau}}$ as a function of the segment averaged shear stress $P_{xy\tau} = \langle P_{xy,i} \rangle_\tau$ for $\tau=0.16$ and $\gamma=0.1$. As can be seen the two curves are essentially linear [11], with very nearly equal slopes. The agreement between the two slopes becomes progressively better as τ increases. The straight line shows the results of a weighted linear least-squares fit to the logarithmic probability ratio data.

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- Transient FR [Evans–Cohen–Morriss '93, Evans–Searles '94+]
- Steady state FR [Gallavotti–Cohen '95]
- FR in stochastic dynamics [Kurchan '98, Lebowitz–Spohn '99]
- FR for Gibbsian measures [Maes '99]



- numerical
- experimental
- theoretical
- mathematical

Including extensions to **quantum** dynamical systems

- More recent reviews: [Rondoni–Mejia–Monasterio '07, Jakšić–P–Rey–Bellet '11]

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- FR are **structural** (model-independent) properties of dynamical systems
- They are refinements of the **2nd Law of Thermodynamics**
- They reduce to **Fluctuation–Dissipation Relations** (Green-Kubo formula, Onsager reciprocity relations) near equilibrium [Gallavotti '96]

Background 2

Effective Quantum Dynamics of Open Systems (Completely Positive Maps)

Small system \mathcal{S}

- Hilbert space $\mathcal{H}_{\mathcal{S}}$ ($\dim < \infty$)
- Hamiltonian $H_{\mathcal{S}}$
- Observables $\mathcal{O} = \mathcal{B}(\mathcal{H}_{\mathcal{S}})$

Large environment \mathcal{E}

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Joint system $\mathcal{S} + \mathcal{R}$

- Hilbert space $\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}}$
- Coupling $V = \sum_a Q_a \otimes R_a \in \mathcal{O} \otimes \mathcal{A}_{\mathcal{E}}$
- Hamiltonian $H = H_{\mathcal{S}} + H_{\mathcal{E}} + \lambda V$

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$$T_t(X) = \text{tr}_{\mathcal{H}_{\mathcal{E}}} \left((\mathbb{1} \otimes \rho_{\mathcal{E}}) e^{itH} (X \otimes \mathbb{1}) e^{-itH} \right)$$

$$T_t : \mathcal{O}_{\mathcal{S}} \rightarrow \mathcal{O}_{\mathcal{S}} \text{ is CP}_1 \quad : \quad \sum_{j,k} Y_j^* T_t(X_j^* X_k) Y_k \geq 0, \quad T_t(\mathbb{1}) = \mathbb{1}$$

$$\text{tr}(\rho T_t(X)) = \text{tr} \left((\rho \otimes \rho_{\mathcal{E}}) e^{itH} (X \otimes \mathbb{1}) e^{-itH} \right)$$

Background 3

Weak Coupling Limit – QDS – Lindbladians

[Davies '74+]

$\{T_t\}_{t \geq 0}$ is **not** a semigroup, but $[H_{\mathcal{E}}, \rho_{\mathcal{E}}] = 0$ and suitable **decay of correlations** in \mathcal{E} imply, for any $\tau > 0$ and $X \in \mathcal{O}$,

$$\lim_{\lambda \rightarrow 0} \sup_{\lambda^2 t \in [0, \tau]} \|T_t(X) - e^{t(\mathcal{L}_S + \lambda^2 \mathcal{L})}(X)\| = 0$$

$\mathcal{L}_S = i[H_S, \cdot]$ and \mathcal{L} (=Davies Generator) generate **QDS** (=CP₁-semigroup).

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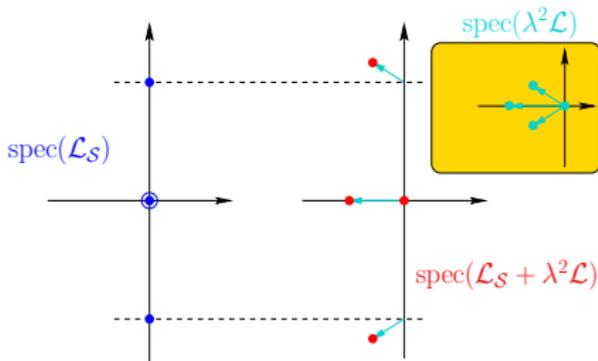
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Remarks.

- For small but finite values of λ
 - [Derezinśki–Jakšić '07] To leading order in λ the resonances of the Liouvillian implementing the dynamics of the joint system $\mathcal{S} + \mathcal{E}$ in the GNS representation are given by $\text{spec}(\mathcal{L}_S + \lambda^2 \mathcal{L})$.
 - [de Roeck '07] Systematic expansion of T_t around its leading contribution $e^{t(\mathcal{L}_S + \lambda^2 \mathcal{L})}$.
- [Derezinśki–de Roeck '08] Extended weak coupling limit.
- Scaling limits in extended systems \implies Boltzmann and diffusion equations
 - [Erdős–Salmhofer–Yau '02+]
 - [de Roeck–Fröhlich–Pizzo '10+]

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[Lindblad, Gorini–Kossakowski–Sudarshan '76]

Structure of QDS-generators (**Lindbladians**)

$$\mathcal{L}(X) = i[T, X] + \Phi(X) - \frac{1}{2}\{\Phi(\mathbb{1}), X\}$$

$T = T^* \in \mathcal{O}$ and $\Phi : \mathcal{O} \rightarrow \mathcal{O}$ a CP-map.

Remark. T and Φ are uniquely determined if one imposes the conditions $\text{tr}(T) = 0$ and $\text{tr}(\Phi) = 0$.

Equilibrium Thermodynamics

Thermal Equilibrium & Detailed Balance

Let ρ be a faithful state on \mathcal{O} (strictly positive density matrix)

Hilbert space structures on \mathcal{O} & dualities

$$\begin{aligned}\langle X|Y\rangle &= \text{tr}(X^* Y) & \langle X|\mathcal{F}(Y)\rangle &= \langle \mathcal{F}^*(X)|Y\rangle \\ \langle X|Y\rangle_\rho &= \text{tr}(\rho X^* Y) & \langle X|\mathcal{F}(Y)\rangle_\rho &= \langle \mathcal{F}^\rho(X)|Y\rangle_\rho\end{aligned}$$

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[Kossakowski–Frigerio–Gorini–Verri '77]

Lindbladian $\mathcal{L} = i[T, \cdot] + \Phi - \frac{1}{2}\{\Phi(\mathbb{1}), \cdot\} \in \text{DB}(\rho)$ if it satisfies **detailed balance** w.r.t. ρ :

$$\mathcal{L}^*(\rho) = 0 \text{ and } \Phi^\rho = \Phi$$

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$\mathcal{L} \in \text{DB}(\rho)$ implies that ρ is a **steady state** of the QDS $e^{t\mathcal{L}}$

$$\text{tr}(\rho e^{t\mathcal{L}}(X)) = \langle \rho|e^{t\mathcal{L}}(X)\rangle = \langle e^{t\mathcal{L}^*}(\rho)|X\rangle = \langle \rho|X\rangle = \text{tr}(\rho X)$$

for all $t \geq 0$ and $X \in \mathcal{O}$.

Equilibrium Thermodynamics

Example: The Davies Generator at Equilibrium

If $\rho_{\mathcal{E}}$ is a thermal equilibrium state at inverse temperature $\beta = (k_B T)^{-1}$ then the Davies generator

$$\mathcal{L}(X) = i[T, X] + \Phi(X) - \frac{1}{2}\{\Phi(\mathbb{1}), X\}$$

satisfies

$$\mathcal{L} \in \text{DB} \left(\frac{e^{-\beta H_S}}{\text{tr}(e^{-\beta H_S})} \right)$$

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$$T = \sum_{a,b} \sum_{\omega \in \text{spec}(i\mathcal{L}_S)} s_{ab}(\omega) Q_a^{(\omega)*} Q_b^{(\omega)}, \quad \Phi(X) = \sum_{a,b} \sum_{\omega \in \text{spec}(i\mathcal{L}_S)} h_{ab}(\omega) Q_a^{(\omega)*} X Q_b^{(\omega)}$$

where

$$h_{ab}(\omega) = 2 \text{Im tr} \left(\rho_{\mathcal{E}} R_a (H_{\mathcal{E}} - \omega - i0)^{-1} R_b \right), \quad s_{ab}(\omega) = \text{Re tr} \left(\rho_{\mathcal{E}} R_a (H_{\mathcal{E}} - \omega - i0)^{-1} R_b \right)$$

$$Q_a^{(\omega)} = P_{\omega}(i\mathcal{L}_S)(Q_a) = \sum_{\mu - \nu = \omega} P_{\nu}(H_S) Q_a P_{\mu}(H_S)$$

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[Spohn '77]

$$\{H_S, Q_a\}' = \mathbb{C}\mathbb{1}, \quad [h_{ab}(\omega)] > 0 \text{ for all } \omega \in \text{spec}(i\mathcal{L}_S)$$

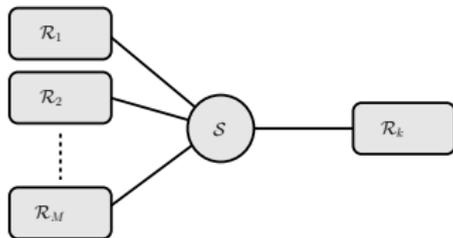
imply that the QDS $e^{t\mathcal{L}}$ is positivity improving. It follows that $\text{Ker}(\mathcal{L}) = \mathbb{C}\mathbb{1}$ and

$$\text{for any state } \rho, \lim_{t \rightarrow \infty} e^{t\mathcal{L}^*}(\rho) = \frac{e^{-\beta H_S}}{\text{tr}(e^{-\beta H_S})} = \text{the unique state in } \text{Ker}(\mathcal{L}^*)$$

Nonequilibrium Thermodynamics

QDS for Open Systems Out of Equilibrium

[Lebowitz-Spohn '78]

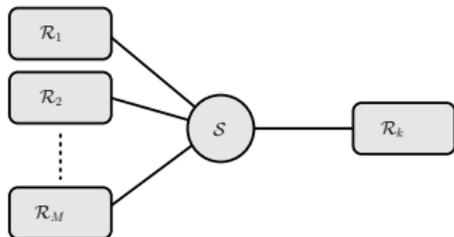


- $\mathcal{E} = \mathcal{R}_1 + \dots + \mathcal{R}_M$
- \mathcal{R}_k in thermal equilibrium at inverse temperature β_k
- $V = \sum_{k,a} Q_{k,a} \otimes R_{k,a}, Q_{k,a} \otimes R_{k,a} \in \mathcal{O} \otimes \mathcal{A}_{\mathcal{R}_k}$
- Davies generator $\mathcal{L} = \sum_k \mathcal{L}_k$
- $\mathcal{L}_k =$ Davies generator for $S + \mathcal{R}_k$
- $\mathcal{L}_k \in \text{DB}(\rho_k = e^{-\beta_k H_S} / \text{tr}(e^{-\beta_k H_S}))$

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- Under Spohn's Conditions $\dim \text{Ker}(\mathcal{L}) = 1$ and $\text{Ker}(\mathcal{L}^*)$ contains a unique **NESS** ρ_+ , s.t. for any state ρ on \mathcal{O}

$$\lim_{t \rightarrow \infty} e^{t\mathcal{L}^*}(\rho) = \rho_+$$

- The **Heat/Entropy flux** observable

$$\phi_k = \mathcal{L}_k(H_S), \quad \psi_k = \mathcal{L}_k(-\log \rho_k) = \beta_k \phi_k$$

describes the energy/entropy current out of heat reservoir \mathcal{R}_k

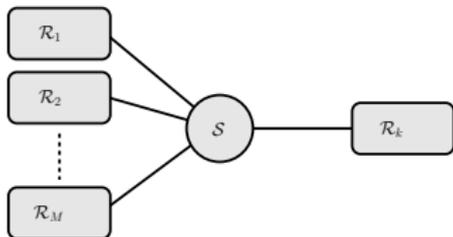
- **1st Law:**

$$\sum_k \text{tr}(\rho_+ \phi_k) = \langle \rho_+ | \mathcal{L}(H_S) \rangle = \langle \mathcal{L}^*(\rho_+) | H_S \rangle = 0$$

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- $\mathcal{L}_k \in \text{DB}(\rho_k = e^{-\beta_k H_S} / \text{tr}(e^{-\beta_k H_S}))$

- The von Neumann entropy $S(\rho) = -\text{tr}(\rho \log \rho)$ satisfies the balance equation

$$\left. \frac{d}{dt} S(e^{t\mathcal{L}^*}(\rho)) \right|_{t=0} = \underbrace{\sigma(\rho)}_{\text{entropy production}} + \underbrace{j(\rho)}_{\text{entropy current}}$$

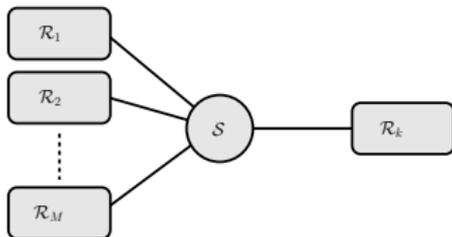
- Entropy current is $j(\rho) = \sum_k \text{tr}(\rho \psi_k)$
- Entropy production $\sigma(\rho)$ is a non-negative lsc convex function of ρ
- In the NESS

$$\sigma(\rho_+) = -j(\rho_+) = -\sum_k \text{tr}(\rho_+ \psi_k) > 0$$

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- $\sigma(\rho) = 0 \iff \beta_1 = \dots = \beta_M = \beta_{\text{eq}}$ and $\rho = \rho_{\text{eq}} = e^{-\beta_{\text{eq}} H_S} / \text{tr}(e^{-\beta_{\text{eq}} H_S})$
- $\beta_1 = \dots = \beta_M = \beta_{\text{eq}} \implies \rho_+ = \rho_{\text{eq}}$
- The Onsager matrix

$$L_{jk} = \left. \frac{\partial}{\partial \beta_k} \text{tr}(\rho_+ \phi_j) \right|_{\beta_1 = \dots = \beta_M = \beta_{\text{eq}}}$$

satisfies Green-Kubo (fluctuation-dissipation) relation

$$L_{jk} = \int_0^\infty \text{tr}(\rho_{\text{eq}} e^{t\mathcal{L}}(\phi_j)\phi_k) dt \quad (j \neq k)$$

- Time reversal invariance implies Onsager's reciprocity relations

$$L_{jk} = L_{kj}$$

Entropic Fluctuations

Let $\mathcal{L} = \sum_k \mathcal{L}_k$ with $\mathcal{L}_k \in \text{DB}(\rho_k)$ and for $\alpha = (\alpha_1, \dots, \alpha_M) \in \mathbb{R}^M$ set

$$\mathcal{L}_\alpha(X) = \sum_k \mathcal{L}_k(X \rho_k^{-\alpha_k}) \rho_k^{\alpha_k}$$

Theorem 1. [de Roeck, Maes, Dereziński, Jakšić–P–Westrich]

Assume that the QDS $e^{t\mathcal{L}}$ is positivity improving (e.g., that Spohn's criterion holds).

- For all $\alpha \in \mathbb{R}^M$, $e^{t\mathcal{L}_\alpha}$ is a positivity improving CP-semigroup.
- For all $\alpha \in \mathbb{R}^M$, $e(\alpha) = \max \text{Re}(\text{spec}(\mathcal{L}_\alpha))$ is a simple eigenvalue of \mathcal{L}_α and its only eigenvalue on the line $\text{Re}(z) = e(\alpha)$.
- For any state ρ there exists a probability measure P_ρ^t on \mathbb{R}^M such that

$$\text{tr} \left(\rho e^{t\mathcal{L}_\alpha}(\mathbb{1}) \right) = \int_{\mathbb{R}^M} e^{-t\alpha \cdot \varsigma} dP_\rho^t(\varsigma)$$

- $e(\alpha)$ is a real-analytic convex function of α and

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \int_{\mathbb{R}^M} e^{-t\alpha \cdot \varsigma} dP_\rho^t(\varsigma) = e(\alpha)$$

holds for α in a complex neighborhood of \mathbb{R}^M .

Entropic Fluctuations

Unraveling – Quantum Trajectories

$$\mathcal{L}_k = \mathcal{L}_{k, \text{no jumps}} + \Phi_k, \quad \mathcal{L}_{k, \text{no jumps}} = i[T_k, \cdot] - \frac{1}{2}\{\Phi_k(\mathbb{1}), \cdot\}$$

Detailed balance ($\mathcal{L}_k \in \text{DB}(\rho_k)$) implies the **spectral decomposition**

$$\Phi_k = \sum_{\omega \in \Omega_k} \Phi_{k, \omega}, \quad \Phi_{k, \omega}(X \rho_k^{-\alpha}) \rho_k^\alpha = e^{-\alpha \omega} \Phi_{k, \omega}(X)$$

with $\Omega_k = \text{spec}(\log \rho_k) - \text{spec}(\log \rho_k)$ (=spectrum of the modular group of ρ_k)

$$\text{Set } \mathcal{L}_{\text{no jumps}} = \sum_k \mathcal{L}_{k, \text{no jumps}}$$

The Dyson expansion of $e^{t\mathcal{L}\alpha}$ around $e^{t\mathcal{L}_{\text{no jumps}}}$ leads to

$$\langle \rho | e^{t\mathcal{L}\alpha}(\mathbb{1}) \rangle = \int e^{-t \sum_j \alpha_j s_j(\xi)} d\mu_\rho^t(\xi)$$

where μ_ρ^t is a probability measure on the set of **Quantum Trajectories** $\xi = [\xi_1, \dots, \xi_N]$

$$N \in \mathbb{N}, \xi_k = (j_k, \omega_k, \mathbf{s}_k), j_k \in \{1, \dots, M\}, \omega_k \in \Omega_k, 0 \leq s_1 \leq \dots \leq s_N \leq t$$

P_ρ^t is the joint distribution of the RV $s_j(\xi) = t^{-1} \sum_{k: j_k=j} \omega_k$ under μ_ρ^t

Entropic Fluctuations

Full Counting Statistics

Let \mathcal{L} be a Davies generator ($\rho_k = e^{-\beta_k \mathcal{H}_S} / \text{tr}(-)$) and $\Pi_{\mathbf{s}}$ denote the joint spectral projection of $\mathbf{S} = (\beta_1 H_{\mathcal{R}_1}, \dots, \beta_M H_{\mathcal{R}_M})$ for **finite reservoirs**. The Positive Operator Valued Measure

$$A \subset \text{spec}(\mathbf{S}) \times \text{spec}(\mathbf{S}) \mapsto \mathcal{P}_A(\cdot) = \sum_{(\mathbf{s}, \mathbf{s}') \in A} \Pi_{\mathbf{s}'} e^{-itH} \Pi_{\mathbf{s}}(\cdot) \Pi_{\mathbf{s}} e^{itH} \Pi_{\mathbf{s}'}$$

describes two sets of subsequent measurements of the reservoir energies separated by a time interval t .

Full Counting Statistics of entropy transport = thermodynamic limit of the measure

$$\mathbb{P}_{\rho}^t(\varsigma) = \text{tr}(\mathcal{P}_{\{(\mathbf{s}, \mathbf{s}') \mid \mathbf{s}' - \mathbf{s} = t\varsigma\}}(\rho \otimes \rho_{\mathcal{R}_1} \otimes \dots \otimes \rho_{\mathcal{R}_M}))$$

The measure P_{ρ}^t of Theorem 1 is the **weak coupling limit** of FCS

$$\int f(\varsigma) dP_{\rho}^t(\varsigma) = \lim_{\lambda \rightarrow 0} \int f(\lambda^{-2}\varsigma) d\mathbb{P}_{\rho}^{t\lambda^{-2}}(\varsigma)$$

Entropic Fluctuations

LDP – CLT

Theorem 1 + Gärtner–Ellis \implies **Large Deviation Principle**

$$P_\rho^t(\Sigma) \simeq \exp\left(-t \inf_{\varsigma \in \Sigma} I(\varsigma)\right) \quad (\Sigma \subset \mathbb{R}^M, t \rightarrow \infty)$$

with rate function

$$I(\varsigma) = - \inf_{\alpha \in \mathbb{R}^M} (\alpha \cdot \varsigma + e(\alpha))$$

Theorem 1 + Bryc \implies **Central Limit Theorem**

$$\lim_{t \rightarrow \infty} P_\rho^t \left(\left\{ \varsigma \mid \sqrt{t}(\varsigma - \langle \varsigma \rangle_\rho^t) \in A \right\} \right) = \mu_D(A)$$

centered Gaussian on \mathbb{R}^M with covariance

$$D_{ij} = \left. \frac{\partial^2 e(\alpha)}{\partial \alpha_i \partial \alpha_j} \right|_{\alpha=0}$$

Entropic Fluctuations

Fluctuation Relations

Theorem 2

If **Time-Reversal Invariance** holds ($\mathcal{L}_k \circ \Theta = \Theta \circ \mathcal{L}_k^{\rho_k}$, $\Theta^*(\rho_k) = \rho_k$) then

$$e(\mathbf{1} - \alpha) = e(\alpha), \quad e(\alpha + \kappa \beta^{-1}) = e(\alpha)$$

with $\mathbf{1} = (1, \dots, 1)$, $\beta^{-1} = (\beta_1^{-1}, \dots, \beta_M^{-1})$ and for any $\alpha \in \mathbb{R}^M$ and $\kappa \in \mathbb{R}$.

- The first identity is the **Evans–Searles symmetry**. It implies $l(-\varsigma) = l(\varsigma) + \mathbf{1} \cdot \varsigma$, i.e.,

$$\frac{P_\rho^t(\varsigma = -\mathbf{s})}{P_\rho^t(\varsigma = \mathbf{s})} \simeq e^{-t \sum_k s_k}$$

- The translation symmetry displayed in the second identity is a consequence of energy conservation [Andrieux et al., '09]. It implies that the rate function enforces the **1st Law**

$$l(\varsigma) = +\infty \text{ unless } \beta^{-1} \cdot \varsigma = 0$$

and that the Gaussian measure μ_D is concentrated on the hyperplane $\beta^{-1} \cdot \varsigma = 0$.

Entropic Fluctuations

Linear Response Theory à la Gallavotti

Theorem 3

- The expected heat/entropy fluxes in the NESS are given by

$$\mathrm{tr}(\rho_+ \phi_j) = \left. \frac{1}{\beta_j} \frac{\partial \mathbf{e}(\boldsymbol{\alpha})}{\partial \alpha_j} \right|_{\boldsymbol{\alpha}=\mathbf{0}}, \quad \mathrm{tr}(\rho_+ \psi_j) = \left. \frac{\partial \mathbf{e}(\boldsymbol{\alpha})}{\partial \alpha_j} \right|_{\boldsymbol{\alpha}=\mathbf{0}} \quad (1)$$

- If time-reversal invariance holds then the Onsager matrix is given by

$$L_{jk} = \left. \partial_{\beta_k} \mathrm{tr}(\rho_+ \phi_j) \right|_{\boldsymbol{\beta}=\boldsymbol{\beta}_{\mathrm{eq}} \mathbf{1}} = - \left. \frac{1}{2\beta_{\mathrm{eq}}^2} \frac{\partial^2 \mathbf{e}(\boldsymbol{\alpha})}{\partial \alpha_k \partial \alpha_j} \right|_{\boldsymbol{\beta}=\boldsymbol{\beta}_{\mathrm{eq}} \mathbf{1}, \boldsymbol{\alpha}=\mathbf{0}} \quad (2)$$

- Combining the two symmetries of Theorem 2, Relation (1) gives

$$\mathrm{tr}(\rho_+ \phi_j) = \frac{1}{\beta_j} (\partial_{\alpha_j} \mathbf{e})(0) = -\frac{1}{\beta_j} (\partial_{\alpha_j} \mathbf{e})(\mathbf{1} - \beta_{\mathrm{eq}} \boldsymbol{\beta}^{-1})$$

and (2) follows from differentiation w.r.t. β_k at $\boldsymbol{\beta} = \beta_{\mathrm{eq}} \mathbf{1}$.

- The Green–Kubo formula follows from direct evaluation of the right hand side of (2).

Main Ingredients of the Proofs of Theorem 1 & 2

Non-Commutative Perron-Frobenius Theory

Theorem 4. \Leftarrow [Evans–Høegh-Krohn '78, Schrader '01]

If the CP semigroup $e^{t\mathcal{L}}$ is positivity improving and $\ell = \max \operatorname{Re}(\operatorname{spec}(\mathcal{L}))$ then

- ℓ is a simple eigenvalue of \mathcal{L} , and the only one on the line $\operatorname{Re}z = \ell$.
- For any state ρ and any non-zero $X \geq 0$

$$\ell = \lim_{t \rightarrow \infty} \frac{1}{t} \log \operatorname{tr} \left(\rho e^{t\mathcal{L}}(X) \right)$$

- If $\mathcal{L}(\mathbb{1}) = 0$ then $\ell = 0$ and there is a faithful state $\rho_+ \in \operatorname{Ker}(\mathcal{L}^*)$ such that

$$\left| \operatorname{tr} \left(\rho e^{t\mathcal{L}}(X) \right) - \operatorname{tr}(\rho_+ X) \right| \leq C e^{-\gamma t} \|X\|$$

for some constants $C, \gamma > 0$, all states ρ and all $X \in \mathcal{O}$.

Structural properties of CP semigroups

- $\mathcal{L}_k \in \operatorname{DB}(\rho_k) \implies \mathcal{L}_k$ commutes with the modular group of ρ_k
- Time reversal invariance $\implies \mathcal{L}_\alpha \circ \Theta = \Theta \circ \mathcal{L}_{1-\alpha}^*$
- $e^{\kappa H_S/2} \mathcal{L}_\alpha(X) e^{\kappa H_S/2} = \mathcal{L}_{\alpha+\kappa\beta-1}(e^{\kappa H_S/2} X e^{\kappa H_S/2})$
- Elementary calculations

Outlook

- $\dim \mathcal{H}_S = \infty$
- Liouvillian approach to FR for FCS in open system with small but finite λ (Jakšić+Panati)
- True Gallavotti-Cohen (steady state) FR