

# Quasi Free Fermionic Systems and Nonequilibrium Steady States Induced by Repeated Interactions

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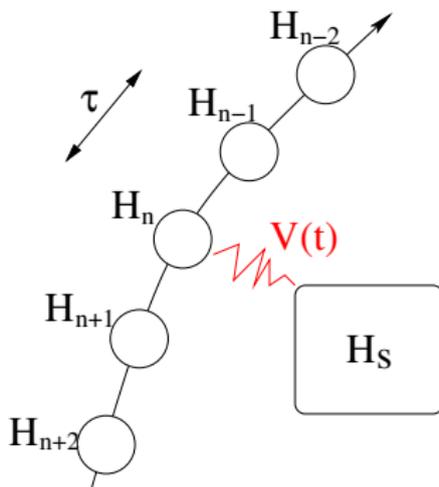
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# Introduction

## The Repeated Interaction Process



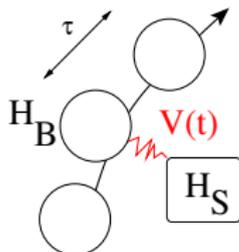
*S. Attal and Y. Pautrat, Ann. Inst. Henri Poincaré 7, 59 (2006).*

## The Repeated Interaction Process

The set up :

- ✓ **System** :  $H_S$
- ✓ **Bath** :  $H_B = \sum_n H_n$   
 $H_n$  Hamiltonian of the  $n^{\text{th}}$  particle
- ✓ **Interaction** :  $V(t)$   
 $V(t) = V_n$  for  $t \in ](n-1)\tau, n\tau]$
- ✓ **Initial State** :  $\rho(0) = \rho_s(0) \otimes \rho_B$   

$$\rho_B(0) = \rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_n \otimes \dots$$



## Time Evolution

Key steps :

✓ Define the operator of evolution  $U(n)$

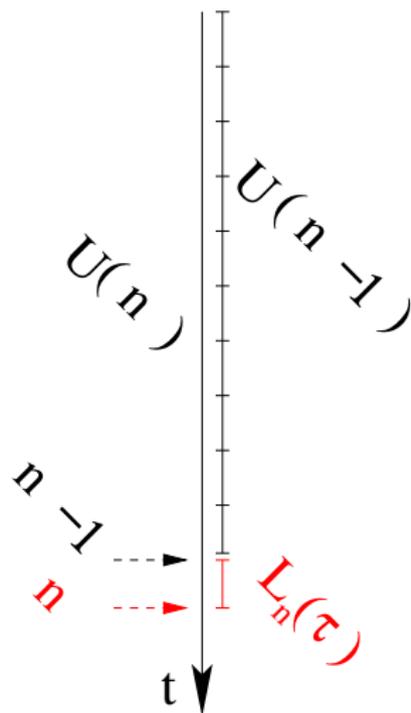
✓ Write  $U(n) = L_n(\tau)U(n-1)$

$$\text{with } L_n(\tau) = K_n(\tau) \otimes \exp(-i\tau \sum_{j \neq n} H_j)$$

$$\text{and } K_n(\tau) = \exp[-i\tau(H_s + V_n + H_n)]$$

✓ Define  $\rho_s(n) = \text{Tr}_B\{\rho(n)\}$

$$\rho(n) = U(n) [\rho(0) \otimes \rho_B(0)] U^\dagger(n)$$



## Reduced density Matrix $\rho_s(t)$

✓ Express

$$\rho_s(n+1) = \text{Tr}_n \{ K_n(\tau) [\rho_s(n) \otimes \rho_n] K_n^\dagger(\tau) \} = \mathcal{L}[\rho_s(n)]$$

$$K_n(\tau) = \exp[-i\tau(H_s + \lambda V_n + H_n)]$$

✓ Take the continuous limit  $\tau \rightarrow 0$

$$\partial_t \rho_s(t) = \mathcal{L}[\rho_s(t)] \quad \mathcal{L}[X] = \lim_{\tau \rightarrow 0} \frac{\mathcal{L}[X] - X}{\tau}$$

Renormalization of the coupling system-bath  $\lambda \rightarrow \lambda/\sqrt{\tau}$

*S. Attal and Y. Pautrat, Ann. Inst. Henri Poincaré 7, 59 (2006).*

✓ Lindblad equation

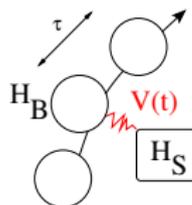
$$\partial_t \rho_s(t) = -i[H_s, \rho_s] - \sum_i \left( \{L_i L_i^\dagger, \rho_s\} - 2L_i \rho_s L_i^\dagger \right)$$

# Considering Fermionic Systems

# The System

A typical Hamiltonian is

$$H_S = \sum_i [c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i] + h \sum_i c_i^\dagger c_i.$$

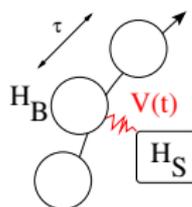


Make it more general :

$$H_S = \sum_{i,j=1}^{L_S} (\mathbf{T}_S)_{i,j} c_i^\dagger c_j \quad \mathbf{T}_S^\dagger = \mathbf{T}_S \quad \{c_i^\dagger, c_j\} = \delta_{i,j}$$

# The Bath

A bath is composed of an infinite number of identical copies  $H_B = \sum_n H_n$ .



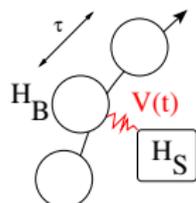
The structure of all copies are identical (same  $H_n$  for all  $n$ )

$$H_n = \sum_{i,j=1}^{L_b} (\mathbf{T}_b)_{i,j} b_{n,i}^\dagger b_{n,j} \quad \mathbf{T}_b^\dagger = \mathbf{T}_b \quad \{b_{i,n}^\dagger, b_{j,m}\} = \delta_{i,j} \delta_{n,m}$$

## The Interaction

During  $]\!n\tau - \tau, n\tau]$  the interaction term

$V(t) = V_n$  is constant.



We choose quadratic interactions between the system and the bath  $c_i^\dagger b_{j,n}$  and  $b_{j,n}^\dagger c_i$ .

$$V_n = \lambda \sum_i^{L_s} \sum_j^{L_b} \left[ \Theta_{i,j} c_i^\dagger b_{j,n} + \Theta_{i,j}^* b_{j,n}^\dagger c_i \right]$$

## Initial State

- ✓ The initial state is **uncorrelated**

$$\checkmark \rho(0) = \rho_S(0) \otimes \rho_B(0)$$

$$\rho_B(0) = \rho_1(0) \otimes \rho_2(0) \otimes \dots \rho_n(0) \otimes \dots$$

- ✓ The initial state of the entire system is **Gaussian**

$$\checkmark \rho_S(0) \propto \exp[-\beta_S H_S] \quad \text{thermalized at temperature } 1/\beta_S$$

$$\checkmark \rho_n(0) \propto \exp[-\beta_b H_n] \quad \text{thermalized at temperature } 1/\beta_b$$

- ✓ Under the time evolution  $\rho(t)$  remains **Gaussian**

- ✓ Importantly  $\rightarrow \rho_s(t)$  remains **Gaussian**

- $\rightarrow$  **Two-points correlators**  $\langle c_j c_i^\dagger \rangle$

## Evolution of the Correlation Matrix

We define the matrix  $\mathcal{M}$  :

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_s & \mathcal{M}_{s,b} \\ \mathcal{M}_{b,s} & \mathcal{M}_b \end{pmatrix} = \begin{pmatrix} \langle c \dots c^\dagger \rangle & \langle c \dots b^\dagger \rangle \\ \langle b \dots c^\dagger \rangle & \langle b \dots b^\dagger \rangle \end{pmatrix}$$

$$\begin{pmatrix} \mathcal{M}_s(n) & 0 \\ 0 & \mathcal{M}_b \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{M}_s(n+1) & \mathcal{M}_{sb} \\ \mathcal{M}_{bs} & \mathcal{M}'_b \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{M}_s(n+1) & 0 \\ 0 & \mathcal{M}_b \end{pmatrix}$$

$$(\mathcal{M}_s)_{i,i} = \langle c_i c_i^\dagger \rangle = 1 - \langle n_s(i) \rangle.$$

$$(\mathcal{M}_b^{(n)})_{i,i} = 1 - \langle n_b(i) \rangle.$$

$$\mathcal{M} = \begin{cases} \mathcal{N} & \text{(density and correlations)} \\ \mathcal{J} & \text{(currents)} \end{cases}$$

## The continuous limit $\tau \rightarrow 0$

In the limit  $\tau \rightarrow 0$  with the renormalisation  $\lambda/\sqrt{\tau} = \Lambda$ , we finally have

$$\partial_t \mathcal{M}_s = -i [ \mathbf{T}_s, \mathcal{M}_s(t) ] - \frac{\Lambda^2}{2} ( \{ \Theta \Theta^\dagger, \mathcal{M}_s(t) \} - 2\Theta \mathcal{M}_b \Theta^\dagger )$$

$$\partial_t \rho_s(t) = -i [ H_s, \rho_s ] - \sum_i ( \{ L_i L_i^\dagger, \rho_s \} - 2L_i \rho_s L_i^\dagger )$$

Holds for arbitrary structure of the interactions  $\mathbf{T}_s$ ,  $\mathbf{T}_b$  and  $\Theta$ .

## Structure of this Equation

Defining  $\mathcal{M}_s = \mathbb{I} - N_s + iJ_s$        $\mathcal{M}_b = \mathbb{I} - N_b + iJ_b$       leads to

$$\partial_t J_s - [\mathbf{T}_s, N_s] + \frac{\Lambda^2}{2} \{ \Theta \Theta^\dagger, J_s \} = \Lambda^2 \Theta J_b \Theta^\dagger,$$

$$\partial_t N_s + [\mathbf{T}_s, J_s] + \frac{\Lambda^2}{2} \{ \Theta \Theta^\dagger, N_s \} = \Lambda^2 \Theta N_b \Theta^\dagger.$$

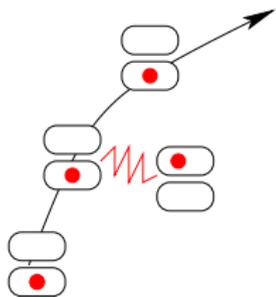
## Equation structure

Defining  $\mathcal{M}_s = \mathbb{I} - N_s + iJ_s$        $\mathcal{M}_b = \mathbb{I} - N_b + iJ_b$       leads to

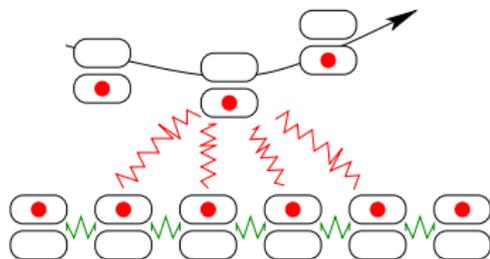
$$\partial_t J_s - [\mathbf{T}_s, N_s] + \frac{\Lambda^2}{2} \{ \Theta \Theta^\dagger, J_s \} = 0,$$

$$\partial_t N_s + [\mathbf{T}_s, J_s] + \frac{\Lambda^2}{2} \{ \Theta \Theta^\dagger, N_s \} = \Lambda^2 \Theta N_b \Theta^\dagger.$$

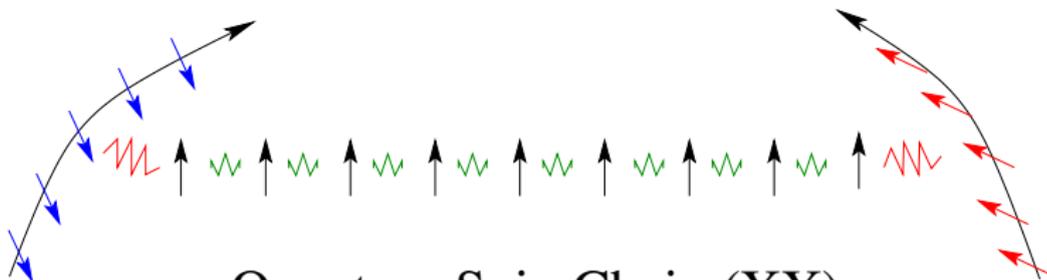
# Concrete Example



Toy Model



Fermionic Chain



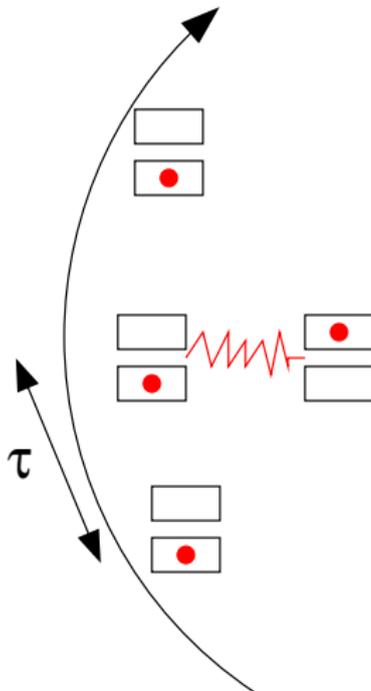
Quantum Spin Chain (XX)

# Toy Model (1)

$$\partial_t n_s(t) + \Lambda^2 n_s(t) = \Lambda^2 n_b,$$

$$n_s(t) = n_b + (n_s(0) - n_b)e^{-t\Lambda^2}.$$

$$n_s^* = n_b$$

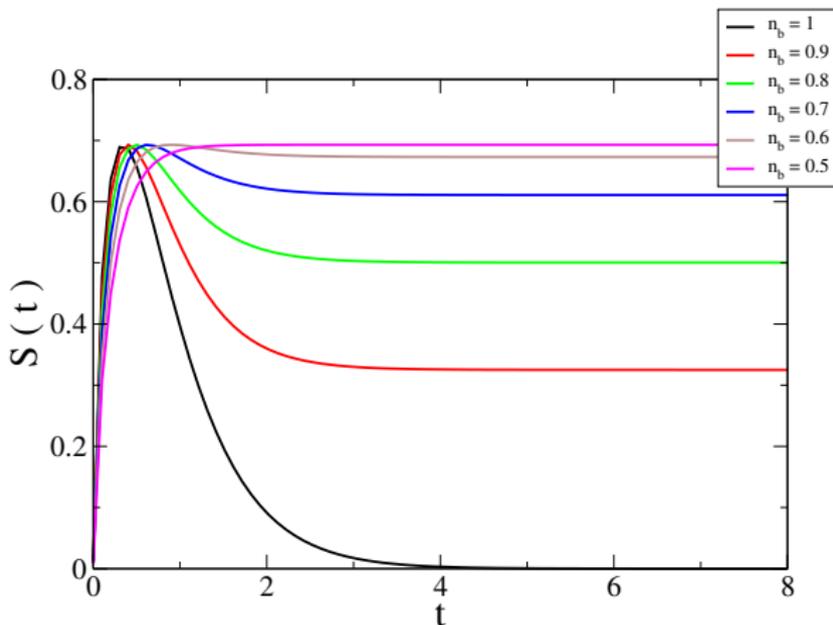


## Toy Model (2)

Entanglement system-environment is given by the von-Neumann Entropy

$$S(t) = -\text{Tr}\{\rho_s \ln(\rho_s)\} = -n_s(t) \ln n_s(t) - [1 - n_s(t)] \ln[1 - n_s(t)].$$

Non-monotonic evolution if  $[n_s(0) - 1/2][n_b - 1/2] < 0$



## Imposing the state of the Bath

Assuming :

- independent fermions in each bath

$$H_n = h \sum_j b_{j,n}^\dagger b_{j,n}$$

- prepared in the same state

$$\langle b_{j,n}^\dagger b_{j,n} \rangle = \langle n_b \rangle \quad \forall j.$$

- for any  $\Theta$

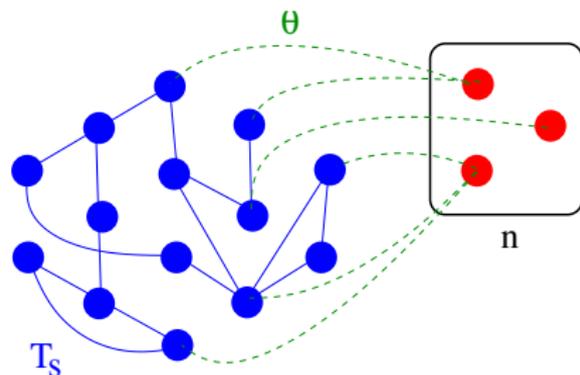
with at least one  $\theta_{i,j} \neq 0$

- for any  $\mathbf{T}_s$

with no disconnected cluster

we show that

$$J_s^* = 0 \quad N_s^* = n_b \mathbb{I}$$

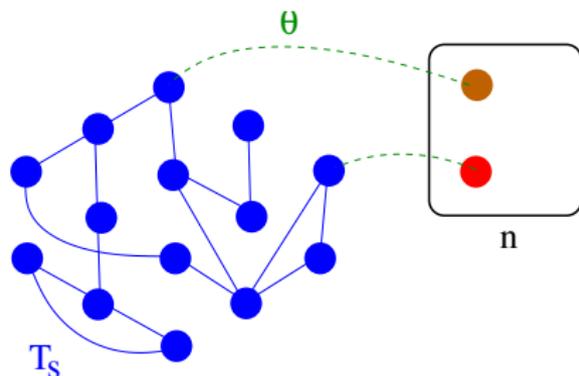


$$(\mathbf{T}_s)_{i,j} [n_s^*(i) - n_s^*(j)] = 0$$

## Two temperatures Bath

For  $(i, j) \neq \text{contact points}$

$$(\mathbf{T}_s)_{i,j} [n_s^*(i) - n_s^*(j)] = 0$$



## Two temperatures Bath

For  $(i,j) \neq \text{contact points}$

$$(\mathbf{T}_s)_{i,j}[n_s^*(i) - n_s^*(j)] = 0$$

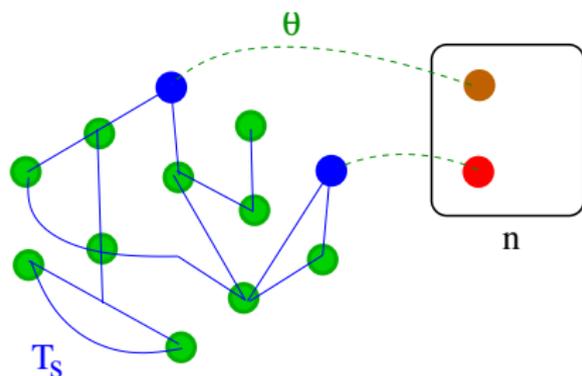
$$J_T(i) = \sum_p (\mathbf{T}_s)_{i,p} (J_s^*)_{i,p} = 0$$

$$J_T(i) = J_+(i) + J_i(i) = 0.$$

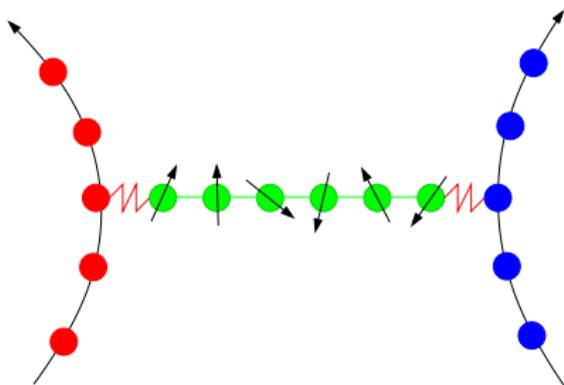
On contact points :  $\gamma$  and  $\gamma'$

$$J_T(\gamma) \propto n_s^* - n_b(1)$$

$$J_T(\gamma') \propto n_s^* - n_b(2)$$



# The Open XX Chain



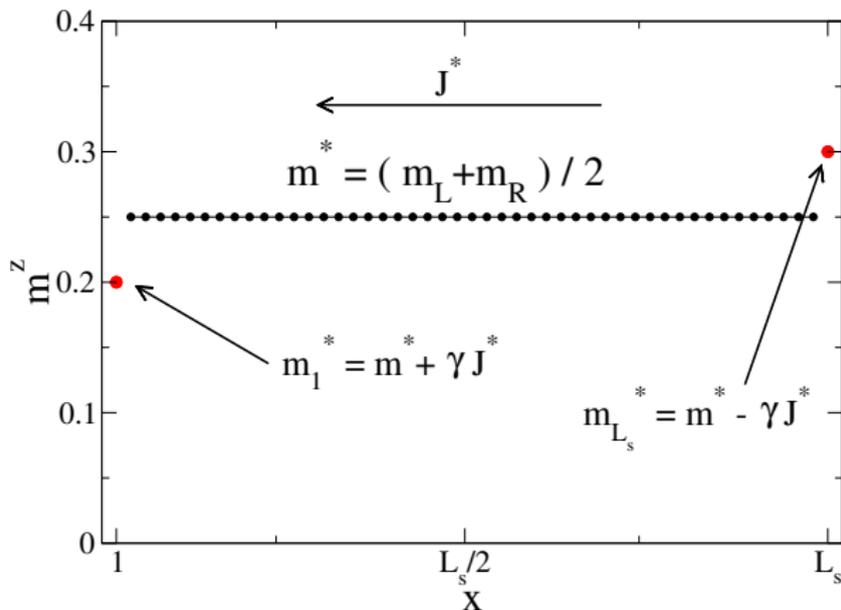
$$H_{XX} = -\frac{\lambda_s}{2} \sum_j \left[ \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y \right] - \frac{h}{2} \sum_j \sigma_j^z$$

$$H_B = \sum_n \left[ H_n^{(L)} + H_n^{(R)} \right] \quad H_n^{(L)} = -h S_{L,n}^z \quad H_n^{(R)} = -h S_{R,n}^z$$

The interactions are  $V(t) = V_n^{(L)} + V_n^{(R)} \quad t \in ]n\tau - \tau, \tau]$

$$V_n^{(L)} = -\frac{\lambda_l}{2} \left[ S_{L,n}^x \sigma_1^x + S_{L,n}^y \sigma_1^y \right], \quad V_n^{(R)} = -\frac{\lambda_l}{2} \left[ \sigma_{L_s}^x S_{R,n}^x + \sigma_{L_s}^y S_{R,n}^y \right]$$

# Stationary State

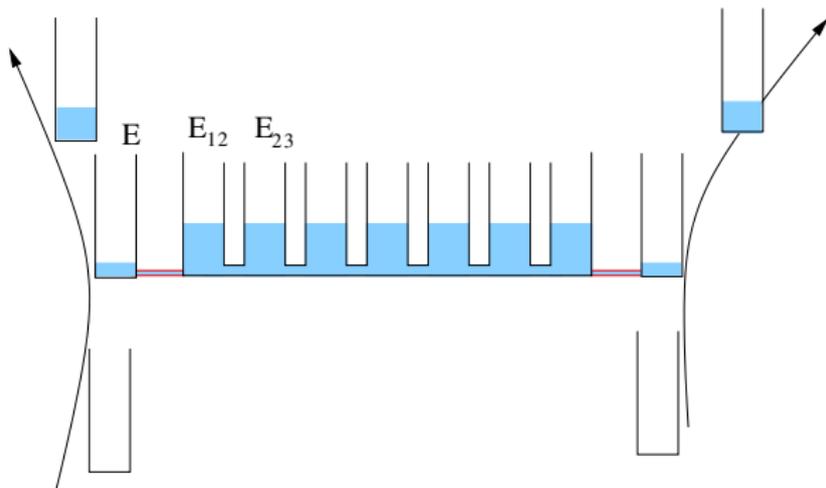


✓ Flat magnetization profil  $m^z(x) = m^*$

✓ Current  $J = (\Delta m^z / 2) \gamma / (1 + \gamma^2)$   $\gamma = \lambda_l^2 / 2\lambda_s$

## Correlation terms

- ✓ The NESS is fully characterized by
- the magnetization profile  $m^z$  and its current  $J^*$
  - all other correlation terms vanish  $\langle c_j^\dagger c_j + c_j^\dagger c_i \rangle = 0$
- ✓ Energy of interaction  $E_{j,j+1} = -\lambda_s \langle c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j \rangle = 0.$



## The reduced density matrix

One can build  $\rho_s^*$

$$\rho_s^* \propto e^{-\sum_l \alpha_l Q_l}, \quad Q_l = \sum_j \left[ c_{j+l}^\dagger c_j + (-1)^l c_j^\dagger c_{j+l} \right] \quad \alpha_l = \alpha_l(m^*, J^*)$$

In the limit  $J^* \ll 1$ , defining

$$H^{(0)} = -(h/2) \sum_j \sigma_j^z \quad g^z = i \sum_j \left[ \sigma_j^x \sigma_{j+1}^y - \sigma_{j+1}^y \sigma_j^x \right]$$

we get

$$\rho_s^* \propto \exp \left[ -\beta_{\text{eff}} H^{(0)} + \frac{2J^*}{1 - (m^*)^2} g^z \right] \quad \beta_{\text{eff}} = \frac{1}{h} \ln \left( \frac{1 - m^*}{1 + m^*} \right)$$

which at high temperature is  $\beta_{\text{eff}} \simeq (\beta_L + \beta_R)/2$ .

W. H. Aschbacher and C. A. Pillet, J. Stat. Phys. 112, 1153 (2003).

## Summary

Using the Repeated Interaction Process,

$$\partial_t \rho_s(t) = -i[H_s, \rho_s] - \sum_i \left( \{L_i L_i^\dagger, \rho_s\} - 2L_i \rho_s L_i^\dagger \right)$$

On quadratic fermionic systems,

$$\partial_t \mathcal{M}_s = -i [\mathbf{T}_s, \mathcal{M}_s(t)] - \frac{\Lambda^2}{2} \left( \{ \Theta \Theta^\dagger, \mathcal{M}_s(t) \} - 2\Theta \mathcal{M}_s \Theta^\dagger \right)$$

For Gaussian state :  $\rho_s(t) \leftrightarrow \mathcal{M}_s(t)$

- ✓ XX Chain : re-construct the reduced density matrix  $\rho_s^*$
- ✓ Considering disordered systems

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- ✓ *can we avoid the destruction of correlations ???*
- ✓ *look for thermalisation conditions ???*

## Thanks to ...

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Thank you for your attention