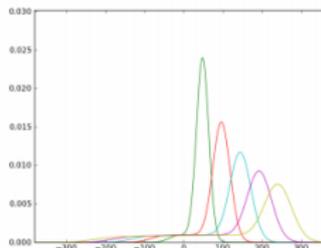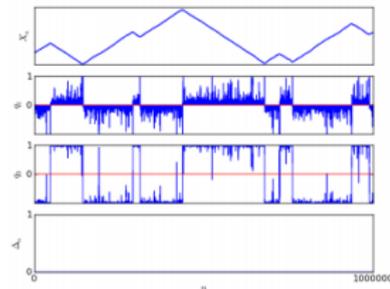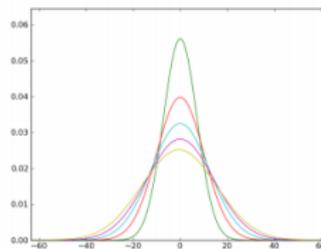# Ballistically induced diffusion in open quantum random walks

Antoine Tilloy, under the supervision of Denis Bernard and Michel Bauer

ENS Paris, Laboratoire de Physique Théorique

July 19, 2013

The main problem of this talk is to understand the regime switch in the simplest OQRW model.
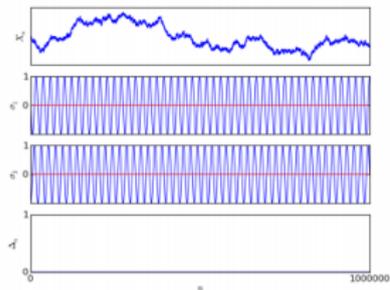
Based *Open quantum random walks* by S. Attal, F. Petruccione C. Sabot and I. Sinayskiy
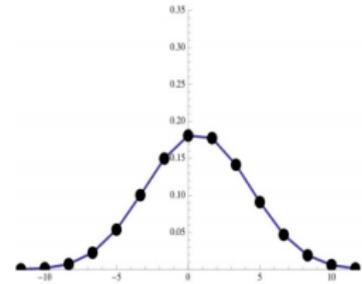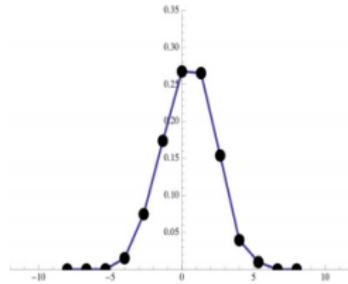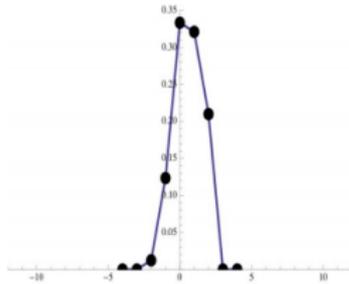
- System Hilbert space: $\mathcal{H}_{internal} \otimes \mathcal{H}_{position} = \mathbb{C}^2 \otimes \mathbb{C}^{\mathbb{Z}}$

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- Completely positive map defined with the help of B_+ and B_-:

$$\rho_0 \otimes |i\rangle\langle i|$$
$$\downarrow$$
$$B_+\rho_0 B_+^{\dagger} \otimes |i+1\rangle\langle i+1| + B_-\rho_0 B_-^{\dagger} \otimes |i-1\rangle\langle i-1|$$

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- Can be dilated on $\mathbb{C}^2 \otimes \mathbb{C}^{\mathbb{Z}} \otimes \left(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes ...\right)$ to get a physical picture.

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After two iterations one gets:

$$B_+^2 \rho_0 {B_+^\dagger}^2 \otimes |i+2\rangle\langle i+2| + \left(B_+ B_- \rho_0 B_-^\dagger B_+^\dagger + B_- B_+ \rho_0 B_+^\dagger B_-^\dagger\right) \otimes |i\rangle\langle i| + B_- \rho_0 B_-^\dagger \otimes |i-2\rangle\langle i-2|$$

After two iterations one gets:

$$B_+^2 \rho_0 B_+^{\dagger\,2} \otimes |i+2\rangle\langle i+2| + \left(B_+ B_- \rho_0 B_-^\dagger B_+^\dagger + B_- B_+ \rho_0 B_+^\dagger B_-^\dagger\right) \otimes |i\rangle\langle i| + B_- \rho_0 B_-^\dagger \otimes |i-2\rangle\langle i-2|$$

So the probabilities to be in $i+2$, i and $i-2$ at time 2 are respectively:

$$Tr\ B_+^2 \rho_0 B_+^{\dagger\,2} \tag{1}$$

$$Tr\ \left(B_+ B_- \rho_0 B_-^\dagger B_+^\dagger + B_- B_+ \rho_0 B_+^\dagger B_-^\dagger\right) \tag{2}$$

$$Tr\ B_-^2 \rho_0 B_-^{\dagger\,2} \tag{3}$$

which cannot be expressed in terms of the previous probabilities only

Tautology

What happens if one measures the trajectory but does not read the result before the end of the experiment ?

Let us imagine that at time t=n, the density matrix of the system is:

$$\rho^n = \sum_i \rho_i |i\rangle\langle i|$$

If the position is measured but not read, the density matrix is now:

$$\frac{1}{Tr(\rho_i)}\rho_i |i\rangle\langle i| \;\; with \; probability \;\; Tr(\rho_i)$$

i.e. by definition of the density matrix, the system is described by the density matrix:

$$\rho^{n+} = \sum_i \frac{Tr(\rho_i)}{Tr(\rho_i)}\rho_i |i\rangle\langle i| = \sum_i \rho_i |i\rangle\langle i| = \rho^n$$

- OQRW are non-Markovian stochastic processes

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- OQRW are classical stochastic processes

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- OQRW are non-Markovian stochastic processes
- OQRW are classical stochastic processes

$\longrightarrow$ Trajectories (or realizations) of OQRW have a sense and can be studied in contrast with what happens with UQW.

Model
Fokker-Planck picture
Stochastic Differential Equation

A trivial Model
Scaling
Fokker-Planck equation
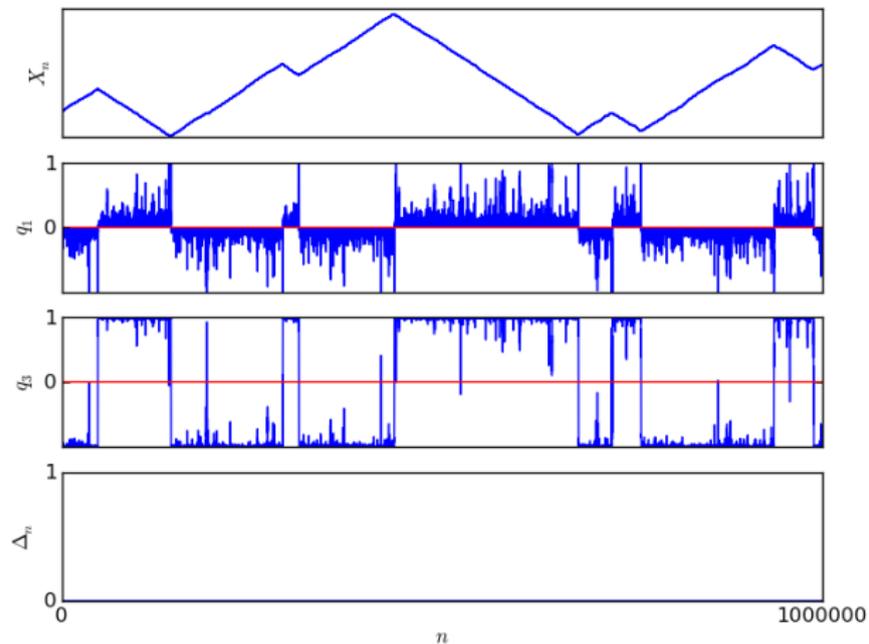
Short Break: study of a toy model

- Reverse-engineer the weird behavior of our OQRW in an even simpler case
- Understand the modification of the diffusion constant

Model
Fokker-Planck picture
Stochastic Differential Equation

A trivial Model
Scaling
Fokker-Planck equation

What would be the Fokker-Planck equation of such an idealized process ?
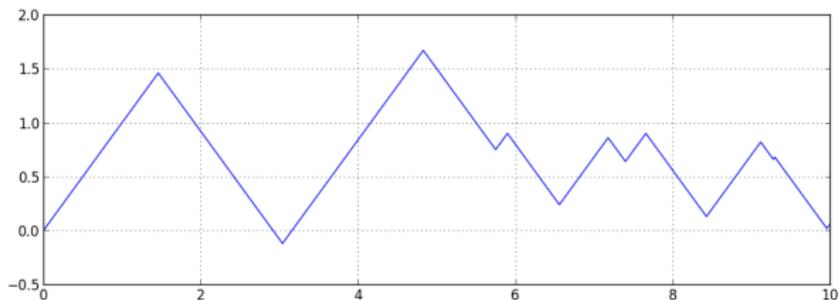
Slope ± 1 with a rate of change λ

Model
Fokker-Planck picture
Stochastic Differential Equation

A trivial Model
Scaling
Fokker-Planck equation

This process if of course non Markovian: we need to define $p_+(x,t)$ and $p_-(x,t)$ the probabilities to be in x at time t with respectively positive and negative slope.

Model
Fokker-Planck picture
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This process if of course non Markovian: we need to define $p_+(x, t)$ and $p_-(x, t)$ the probabilities to be in x at time t with respectively positive and negative slope.

$$\partial_t p_+ + \partial_x p_+ - \lambda(p_+ - p_-) = 0$$
$$\partial_t p_- - \partial_x p_- - \lambda(p_- - p_+) = 0$$

(4)

Model
Fokker-Planck picture
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This process if of course non Markovian: we need to define $p_+(x, t)$ and $p_-(x, t)$ the probabilities to be in x at time t with respectively positive and negative slope.

$$\partial_t p_+ + \partial_x p_+ - \lambda(p_+ - p_-) = 0 \\ \partial_t p_- - \partial_x p_- - \lambda(p_- - p_+) = 0 \tag{4}$$

i.e.

$$\partial_t P = -\sigma_z \partial_x P + \lambda(\mathbb{I} - \sigma_x)P \tag{5}$$

Model
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Results:

Model
Fokker-Planck picture
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Results:

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- No need for a Laplacian term to converge to a Gaussian at large time
- The matrix σ_x couples the two ballistic evolution in the Fokker-Planck picture and allows the regime switch.

Model
Fokker-Planck picture
Stochastic Differential Equation

A trivial Model
Scaling
Fokker-Planck equation

We want to find the most general scaling limit to be sure not to forget anything at the continuous limit. We try to find B_+ and B_- in the following form:

$$
\begin{aligned}
B_+ &= \frac{1}{\sqrt{2}} \left(\mathbb{I} + \sqrt{\epsilon} N_+ + \epsilon M_+ + o(\epsilon) \right) \\
B_- &= \frac{1}{\sqrt{2}} \left(\mathbb{I} + \sqrt{\epsilon} N_- + \epsilon M_- + o(\epsilon) \right)
\end{aligned}
\tag{6}
$$

And we write the full density matrix:

$$
\rho = \int dx \, \rho(x, t) \, |x\rangle\langle x| \tag{7}
$$

Model
Fokker-Planck picture
Stochastic Differential Equation

A trivial Model
Scaling
Fokker-Planck equation

Then we keep on the dirty way writing:

$$\rho(x, t + dt) = B_- \rho(x + dx, t) B_-^\dagger + B_+ \rho(x - dx, t) B_+^\dagger$$

And the previous expression for the B's gives:

$$
\begin{aligned}
\frac{\partial}{\partial t} \rho(x, t) =& \frac{1}{dt} \left(-\rho(x, t) + \frac{\rho(x + dx, t) + \rho(x - dx, t)}{2} \right) \\
&+ \frac{\sqrt{\epsilon}}{2dt} \left(N_+ \rho(x - dx, t) + N_- \rho(x + dx) + \rho(x - dx, t) N_+^\dagger + \rho(x + dx) N_-^\dagger \right) \\
&+ \frac{\epsilon}{2dt} (...)
\end{aligned}
$$

Model
Fokker-Planck picture
Stochastic Differential Equation

A trivial Model
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Fokker-Planck equation

Constraint:

- keep the maximum number of non diverging terms
- verify $B_+^\dagger B_+ + B_-^\dagger B_- = \mathbb{I}$ up to order ϵ

This gives:

Scaling

$$B_+ = \frac{1}{\sqrt{2}} \left(\mathbb{I} + \sqrt{\epsilon} N + \epsilon \left(iH_+ - S_+ \right) + o(\epsilon) \right)$$

$$B_- = \frac{1}{\sqrt{2}} \left(\mathbb{I} - \sqrt{\epsilon} N + \epsilon \left(iH_- - S_- \right) + o(\epsilon) \right) \qquad (8)$$

$$\text{with} \quad S_+ + S_- = N^\dagger N$$

Model
Fokker-Planck picture
Stochastic Differential Equation

A trivial Model
Scaling
Fokker-Planck equation

Writing $H = \frac{H_+ + H_-}{2}$ and $dx^2 = \epsilon = dt$ one gets eventually:

Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\triangle\rho - \left(N\frac{\partial \rho}{\partial x} + \frac{\partial \rho}{\partial x}N^\dagger\right) + i\left[H, \rho\right] + N^\dagger\rho N - \left\{\frac{NN^\dagger}{2}, \rho\right\} \quad (9)$$

Which shows a convective term similar to what one can see in the trivial model.

Model
Fokker-Planck picture
Stochastic Differential Equation

Derivation
Back to Fokker-Planck
Example

We want to use the trajectory representation we described at the beginning to understand the regime switch better.

Let us write ρ_n the partial density matrix once position has been measured at time n and X_n the corresponding position.

Model
Fokker-Planck picture
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Example

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Let us write ρ_n the partial density matrix once position has been measured at time n and X_n the corresponding position.

σ_{n+1} the random variable $\mathcal{F}_{n+1}$ mesurable equal to $+1$ with probability $tr(B_+ \rho_n B_+^\dagger)$ and -1 with the complementary probability.

Model
Fokker-Planck picture
Stochastic Differential Equation

Derivation
Back to Fokker-Planck
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σ_{n+1} the random variable $\mathcal{F}_{n+1}$ mesurable equal to $+1$ with probability $tr(B_+\rho_n B_+^\dagger)$ and -1 with the complementary probability.

$$\rho_{n+1} = \frac{B_+\rho_n B_+^\dagger}{tr(B_+\rho_n B_+^\dagger)}\mathbb{I}_{(\sigma_{n+1}=+1)} + \frac{B_-\rho_n B_-^\dagger}{tr(B_-\rho_n B_-^\dagger)}\mathbb{I}_{(\sigma_{n+1}=-1)} \quad (10)$$

$$X_{n+1} = X_n + \mathbb{I}_{(\sigma_{n+1}=+1)} - \mathbb{I}_{(\sigma_{n+1}=-1)} \quad (11)$$

Model
Fokker-Planck picture
Stochastic Differential Equation

Derivation
Back to Fokker-Planck
Example

This gives:

$$\rho_{n+1} - \rho_n = \frac{1}{2} \left(\frac{B_+ \rho_n B_+^\dagger}{tr(B_+ \rho_n B_+^\dagger)} + \frac{B_- \rho_n B_-^\dagger}{tr(B_- \rho_n B_-^\dagger)} - 2\rho_n \right)$$
$$+ \frac{1}{2} \left(\frac{B_+ \rho_n B_+^\dagger}{tr(B_+ \rho_n B_+^\dagger)} - \frac{B_- \rho_n B_-^\dagger}{tr(B_- \rho_n B_-^\dagger)} \right) (X_{n+1} - X_n) \tag{12}$$

Then inserting the ϵ-development one gets:

$$\rho_{n+1} - \rho_n = \epsilon \left(\frac{i}{2}[H, \rho_n] + N\rho_n N^\dagger - \frac{1}{2}\{N^\dagger N, \rho_n\} - 2\Re(tr(N\rho_n)) \left(N\rho_n + \rho_n N^\dagger \right) + 4\Re(tr(N\rho_n))^2 \rho_n \right)$$
$$+ \sqrt{\epsilon} \left(N\rho_n + \rho_n N^\dagger - 2\Re \left[tr(N\rho_n) \right] \rho_n \right) (X_{n+1} - X_n)$$
$$X_{n+1} - X_n = \pi_{n+1} + 2\sqrt{\epsilon}\Re \left[tr(N\rho_n) \right] \tag{13}$$

Where $\pi_n = X_n - \mathbb{E}(X_n | \mathcal{F}_{n-1})$

Model
Fokker-Planck picture
Stochastic Differential Equation

Derivation
Back to Fokker-Planck
Example

Eventually with the notations:

$$H = \frac{H_+ + H_-}{2}$$
$$\mathcal{L}(\rho) = N\rho N^\dagger - \frac{1}{2}\{N^\dagger N, \rho\}$$
$$D(\rho) = N\rho + \rho N^\dagger - 2\Re[tr(N\rho)]\rho$$
$$U(\rho) = 2\Re[tr(N\rho)]$$

We get:

Stochastic differential equation

$$d\rho_t = \{i[H, \rho_t] + \mathcal{L}(\rho_t)\} \, dt + D(\rho_t)dW_t$$
$$dX_t = dW_t + U(\rho_t)dt \tag{14}$$

Model
Fokker-Planck picture
Stochastic Differential Equation

Derivation
Back to Fokker-Planck
Example

A good consistency check is to go back to the Fokker-Planck equation using the equivalence of the two approaches:

$$\int dx \rho(x, t)|x\rangle\langle x| = \mathbb{E}[\rho_t|X_t\rangle\langle X_t|]$$

Model
Fokker-Planck picture
Stochastic Differential Equation

Derivation
Back to Fokker-Planck
Example

A good consistency check is to go back to the Fokker-Planck equation using the equivalence of the two approaches:

$$\int dx \rho(x, t)|x\rangle\langle x| = \mathbb{E}[\rho_t |X_t\rangle\langle X_t|]$$

Both sides are distributions: make them act on f^*g and differentiate. Then $\partial_t \rho(x, t)$ will simply be found by duality.

$$d(\int dx \hat{\rho}(x, t) f^*(x) g(x)) = \mathbb{E}[d(\rho_t \langle f|X_t\rangle\langle X_t|g\rangle)]$$

Model
Fokker-Planck picture
Stochastic Differential Equation

Derivation
Back to Fokker-Planck
Example

Then one just has to compute.

- By Itô's Lemma one has:

$$
d(\langle f|X_t\rangle\langle X_t|g\rangle) = \partial(f^*g)dW_t + \left[U(\rho_t)\partial(f^*g) + \frac{1}{2}\partial^2(f^*g)\right]
$$
(15)

- Then

$$
\begin{aligned}
d\Big(\int dx\hat{\rho}(x,t)f^*(x)g(x)\Big) &= \mathbb{E}[d(\rho_t)\langle f|X_t\rangle\langle X_t|g\rangle + \rho_t d(\langle f|X_t\rangle\langle X_t|g\rangle) + d(\rho_t)d(\langle f|X_t\rangle\langle X_t|g\rangle)] \\
&= \mathbb{E}(\{i[H,\rho_t] + \mathcal{L}(\rho_t)\} f^*(X_t)g(X_t) + \\
&\qquad \rho_t\left[U(\rho_t)\partial(f^*g)(X_t) + \frac{1}{2}\partial^2(f^*g)(X_t)\right] + D(\rho_t)\partial(f^*g)\ )dt
\end{aligned}
$$

Model
Fokker-Planck picture
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Back to Fokker-Planck
Example

Eventually one finds, as expected:

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\triangle\rho - \left(N\frac{\partial \rho}{\partial x} + \frac{\partial \rho}{\partial x}N^{\dagger}\right) + i\left[H, \rho\right] + N^{\dagger}\rho N - \left\{\frac{NN^{\dagger}}{2}, \rho\right\} \tag{16}$$

Model
Fokker-Planck picture
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Example: Let us go back to our example. We restrict ourselves to real matrices $iH = i\sigma_2$ and $N = a\sigma_3$. Hence ρ_t can be written the following way:

$$\rho_t = \frac{1}{2}(\mathbb{I} + q_1\sigma_1 + q_3\sigma_3)$$

with $q_1^2 + q_3^2 \leq 1$ The SDE gives:

$$
\begin{aligned}
dq_3 &= q_1 dt + 2a(1 - q_3^2)dW_t \\
dq_1 &= -2(q_3 + a^2 q_1)dt - 2aq_1 q_3 dW_t
\end{aligned}
\tag{17}
$$

Model
Fokker-Planck picture
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Using state purification one can reduce further the number of parameters to one angle θ which verifies the following SDE:

$$d\theta_t = -2\left(1 + a^2 \cos\theta_t \sin\theta_t\right) dt - \boxed{2a\sin\theta_t}\, dW_t \qquad (18)$$

Still not a canonical form to apply Kramers' results $\longrightarrow$ change of variable: $y_t = -|\log\tan\theta_t/2|$ to get a SDE of the form:

$$dy_t = -V'(y_t) + 2a\, dW_t$$

With $V(y) = -2\left(\pm\sinh(y) + 2a^2 \log\cosh(y)\right)$

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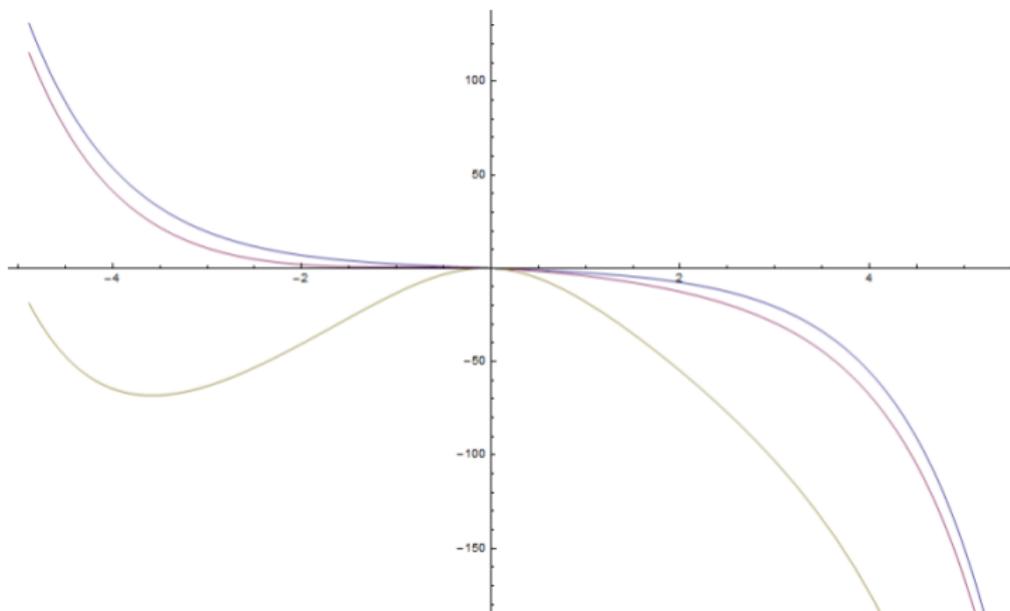
Figure : Potential $V(y) = -2\left(\pm \sinh(y) + 2a^2 \log \cosh(y)\right)$ for $a = 0.25$, $a = 1$ and $a = 3$

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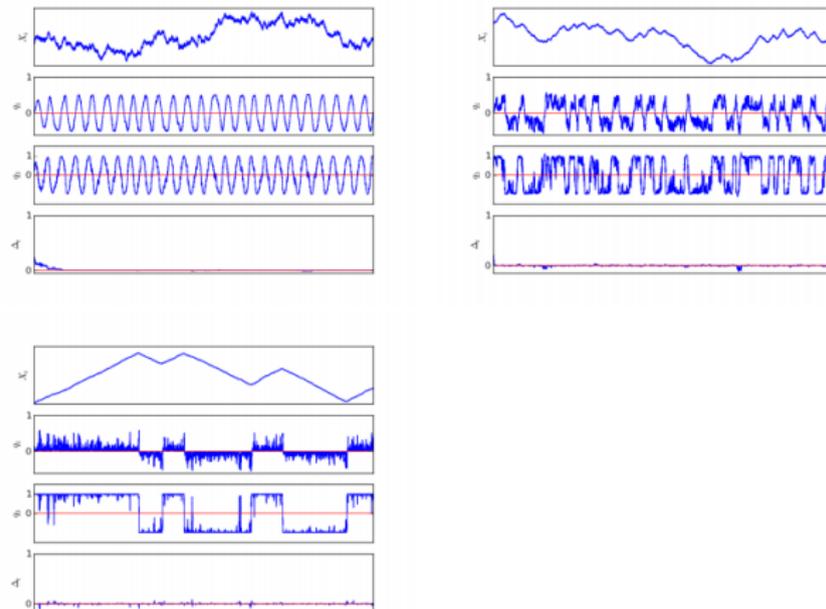
Figure : Illustration of the 3 regimes, for $a = 0.25$ *oscillating*, $a = 1$ *critical* and $a = 3.0$ *ballistic*

Model
Fokker-Planck picture
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Eventually, one can use Kramers' results (basically the relation between kinetics and activation energy) for large a to get:

$$< \tau > \simeq e^{\frac{\Delta V}{4a^2}} \simeq a^2$$

Then the behavior for X_t is trivial because:

$$dX_t = dW_t + 2\Re \left(tr N \rho_t \right) dt \simeq 2a(-1)^{N_t} dt$$

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"Philosophical" summary of the talk

- Trajectories of OQRW are as simple as trajectories of classical processes and are therefore a complementary way to study OQRW.

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"Philosophical" summary of the talk

- Trajectories of OQRW are as simple as trajectories of classical processes and are therefore a complementary way to study OQRW.
- OQRW can give rise to very high diffusion constants with small noise terms.

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"Philosophical" summary of the talk

- Trajectories of OQRW are as simple as trajectories of classical processes and are therefore a complementary way to study OQRW.

- OQRW can give rise to very high diffusion constants with small noise terms.

- Counting processes can emerge from Gaussian fluctuations. There is no need to have a counting process in a Belavkin equation to see jumps.

Model
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Next on the theoretical side:

- Exhaustively explore the space of B matrices at least at the continuous limit.

- Investigate higher dimensions (both in internal space and position space).

- Include temperature in this description.

- Add disorder with random B matrices to see if Anderson localization breaks down with OQRW (idea of Alain Joye)

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On the application side

- Find situations with unexpectedly high diffusion constants
- Find applications of the classical process to other fields (Biology for example).

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For more information:

- An unreadable PRL failure that is on the arxiv

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Thank you for your attention.