Bound States and Recurrence Properties of Quantum Walks

Autrans
18.07.2013

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Joint work with:
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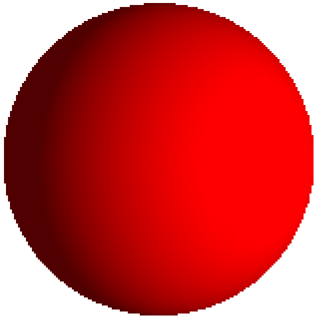
Alberto F. Grünbaum (Berkeley)
Luis Velázquez (Zaragoza)

What are Quantum Walks?

Dynamics of

- a single particle



What are Quantum Walks?

Dynamics of

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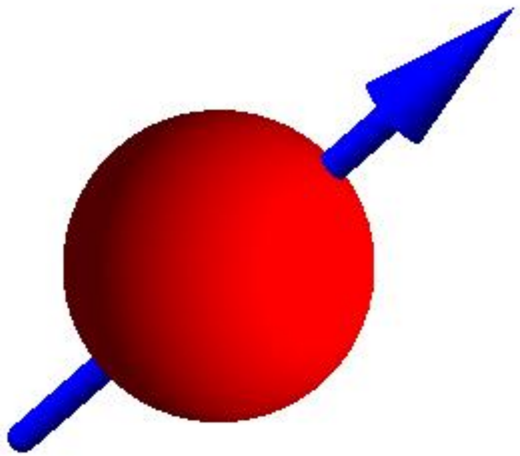
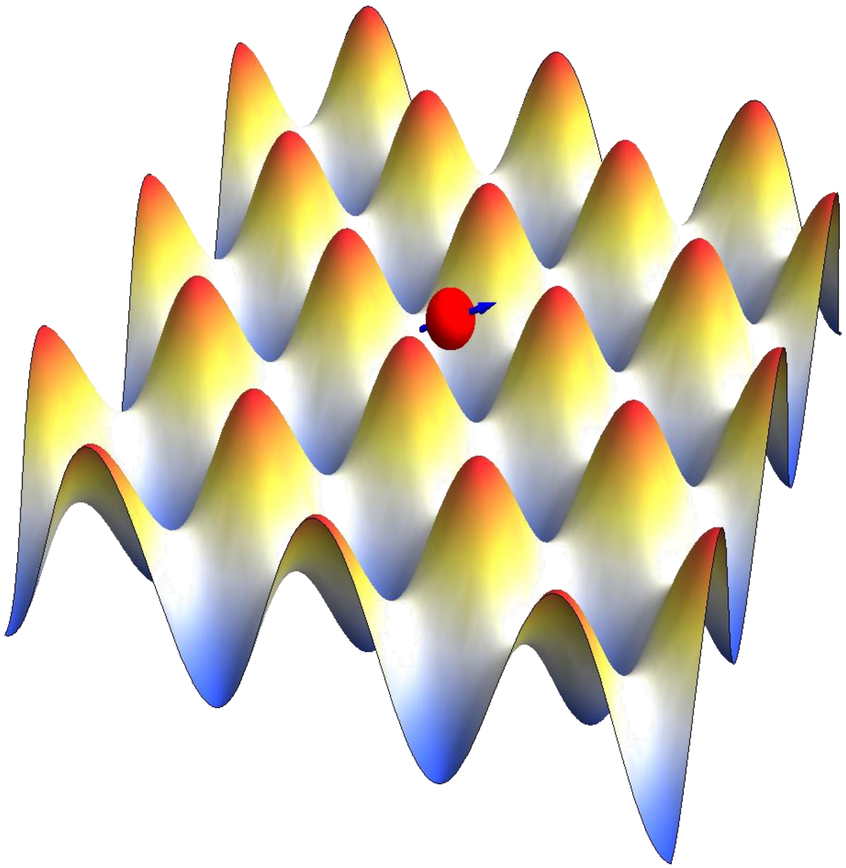
- with internal degree of freedom



What are Quantum Walks?

Dynamics of

- a single particle
- with internal degree of freedom
- on a lattice



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Dynamics of

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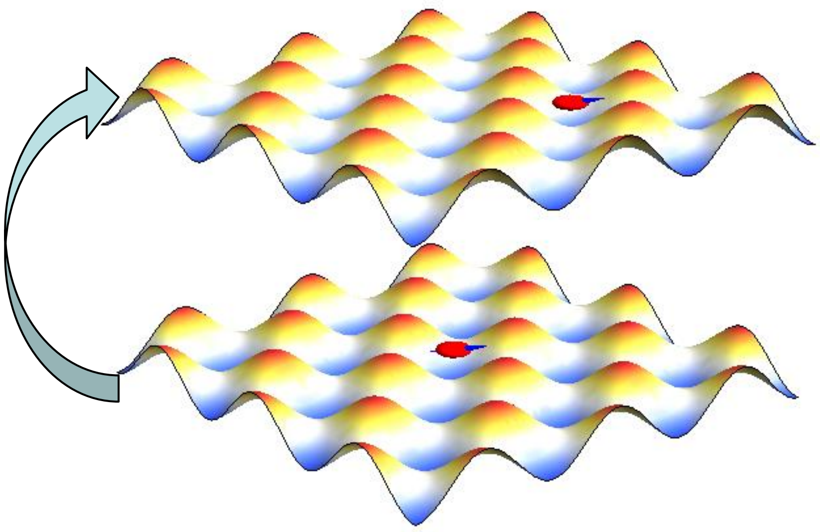
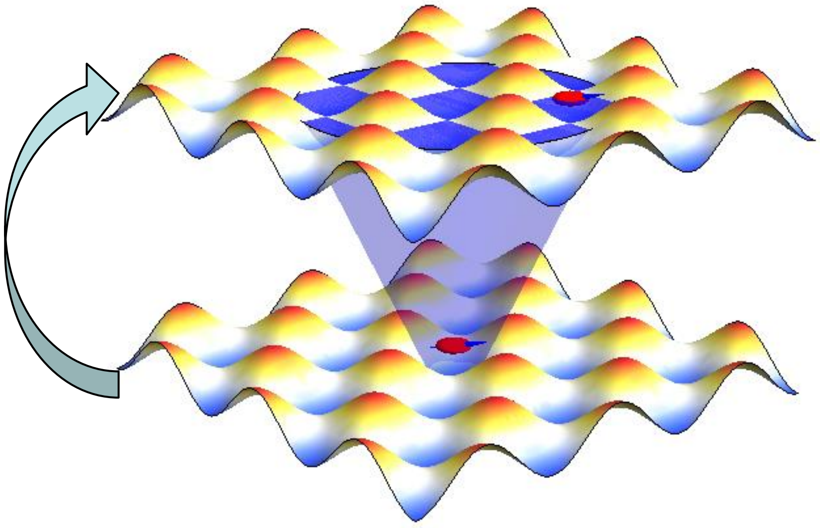
- with internal degree of freedom
- on a lattice
- in discrete timesteps



What are Quantum Walks?

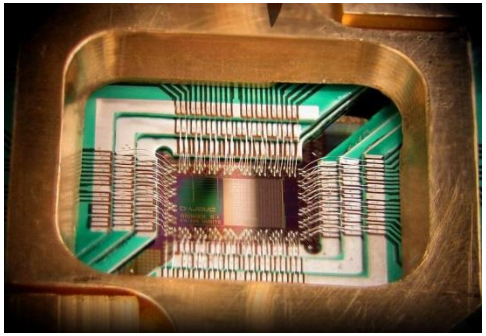
Dynamics of

- a single particle
- with internal degree of freedom
- on a lattice
- in discrete timesteps
- strictly local

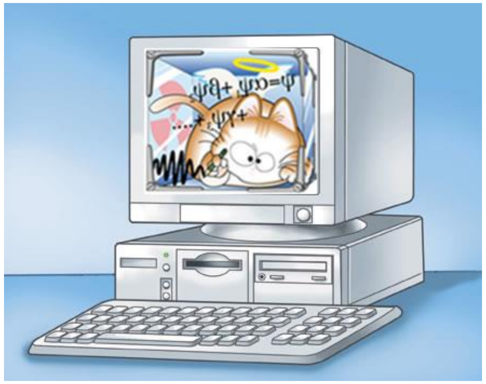


Why?

- Step towards quantum-simulators
    - Simulation of lattice systems in discrete time steps
    - Simulation of one particle-effects
    - Quantum Biology
- “quantization” of random walks
    - Searching in graphs
    - Quantum computer

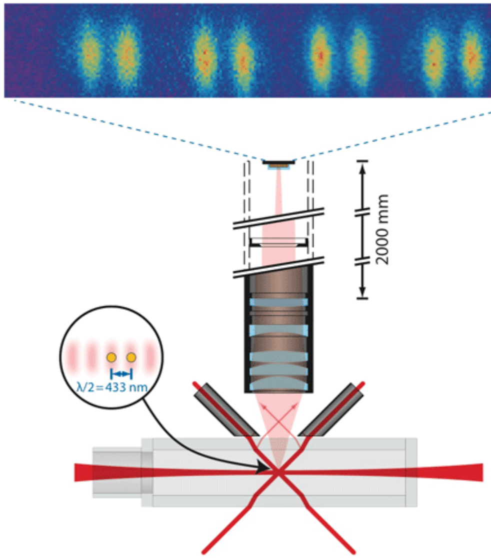


source: wikipedia.org



source: ucm.es

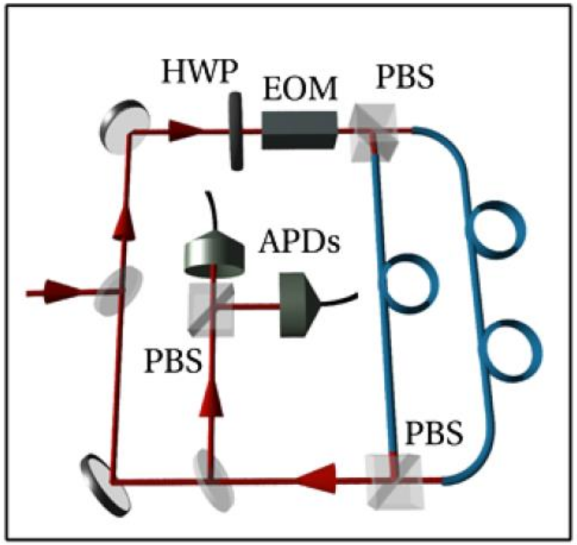
Experimental Realisations



source: iap.uni-bonn.de/

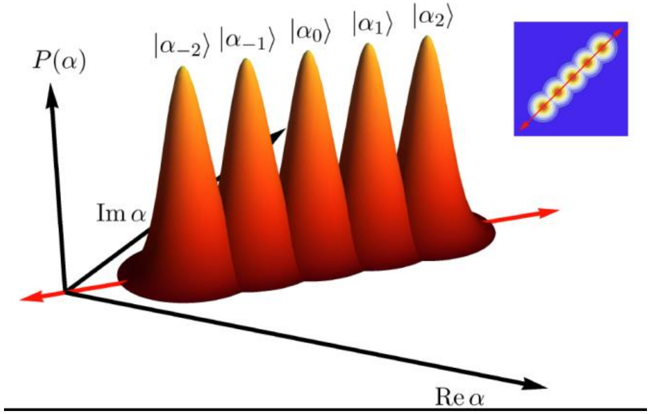
atom in optical lattice

optical fibres



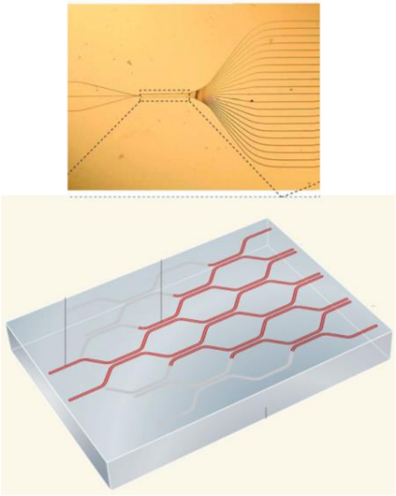
source: Schreiber et al. (2011)

phase space of trapped ions



source: Matjeschk et al. (2012)

wave guide arrays



source: Peruzzo et al. (2010)

Outline

- Propagation properties
- Bound states in interacting quantum walks
- Recurrence properties

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- Bound states in interacting quantum walks
- Recurrence properties

1D Example: Coined Quantum Walk

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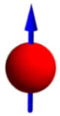
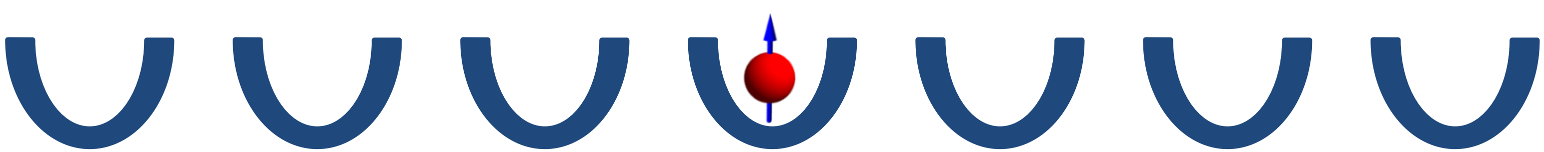


$$\mathbb{C}^2$$

1D Example: Coined Quantum Walk



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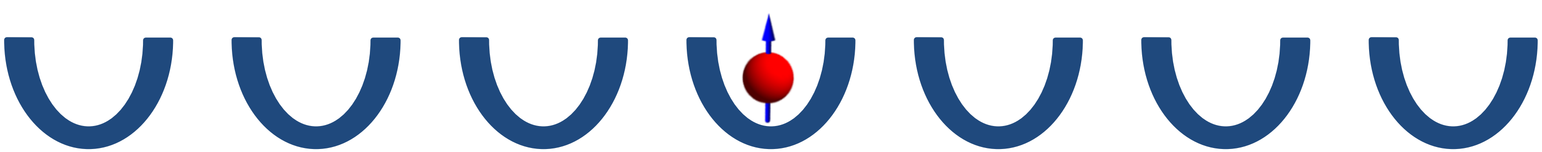

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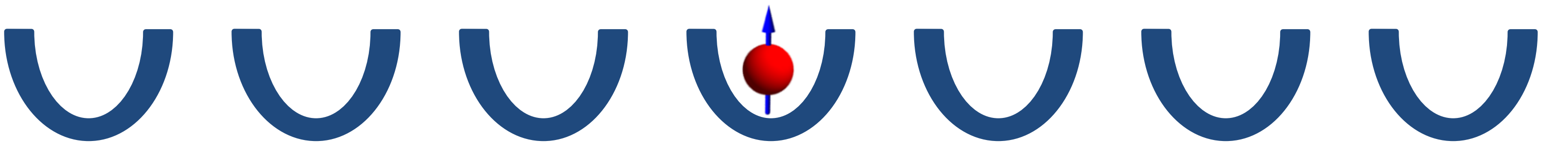
Hilbert space: $\mathcal{H} = \ell_2(\mathbb{Z}) \otimes \mathbb{C}^2$

1D Example: Coined Quantum Walk

- Basis: $|x, \pm\rangle$



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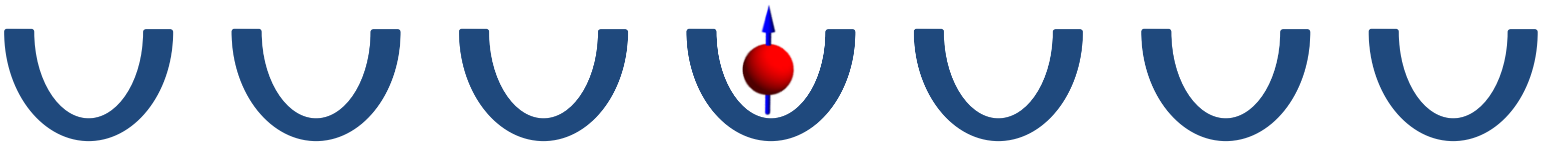
  $$W$$



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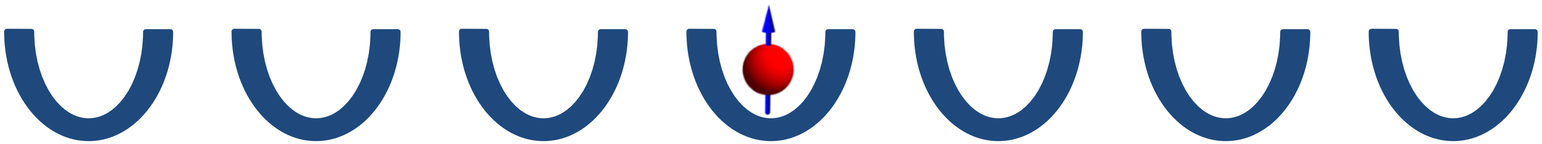
- Time evolution:

$$\phi_t = W^t \phi_0$$



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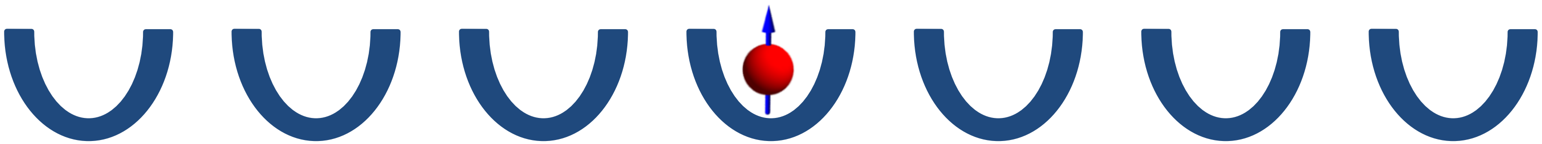

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$$U = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$$

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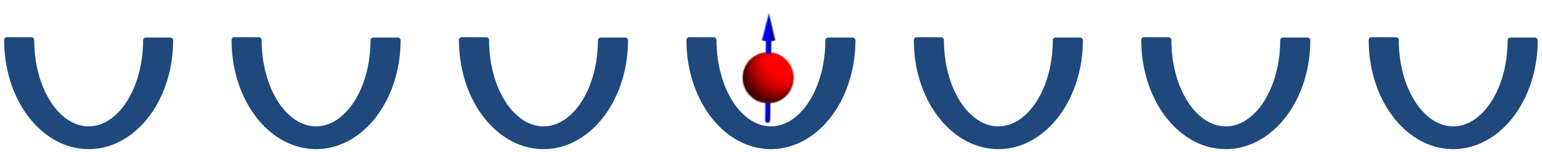

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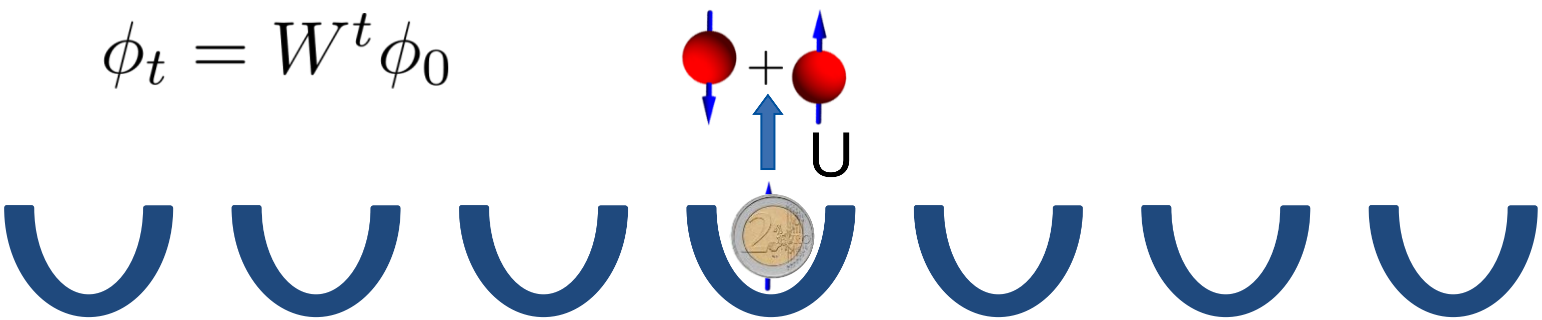

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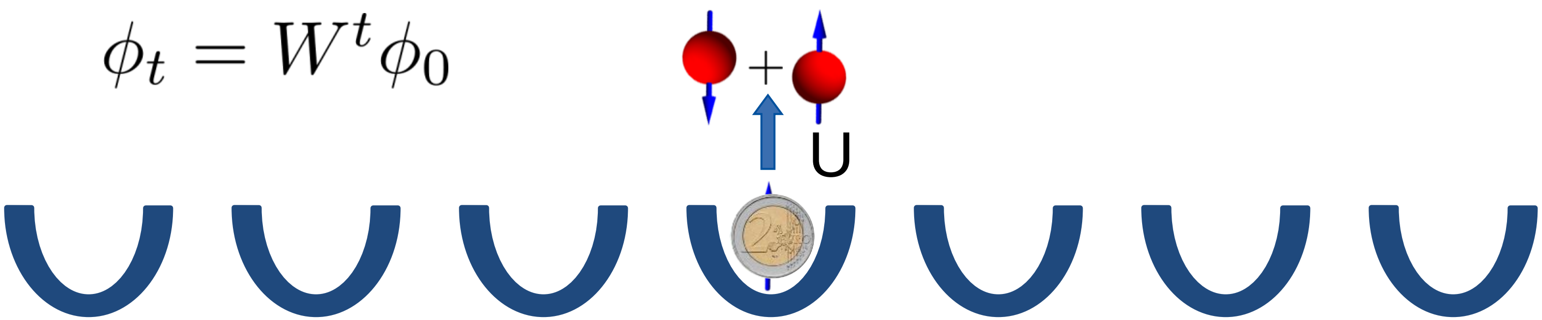

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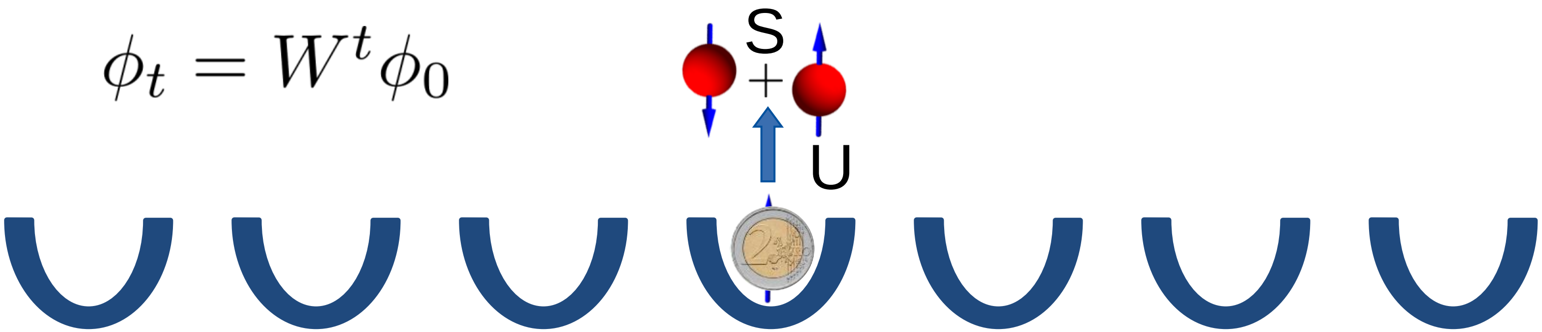

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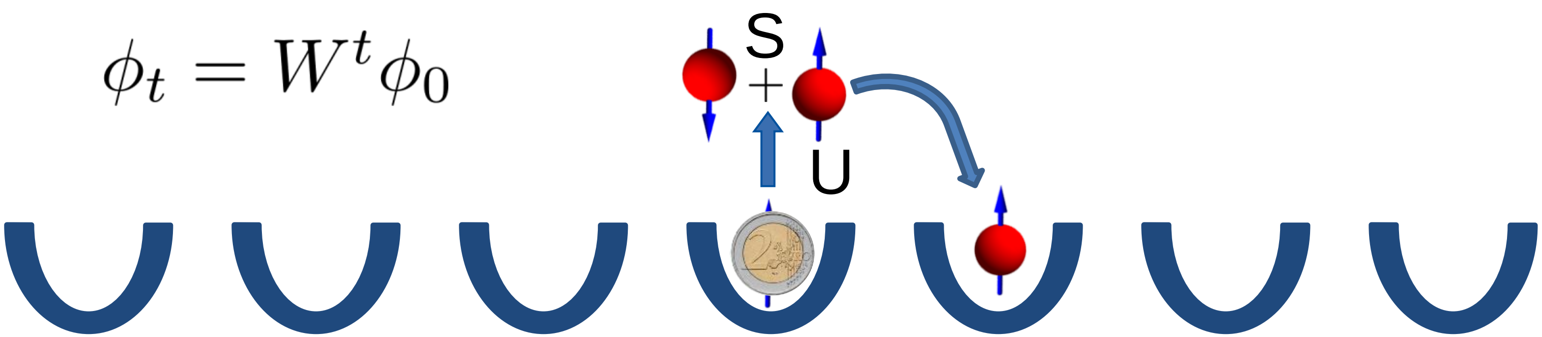

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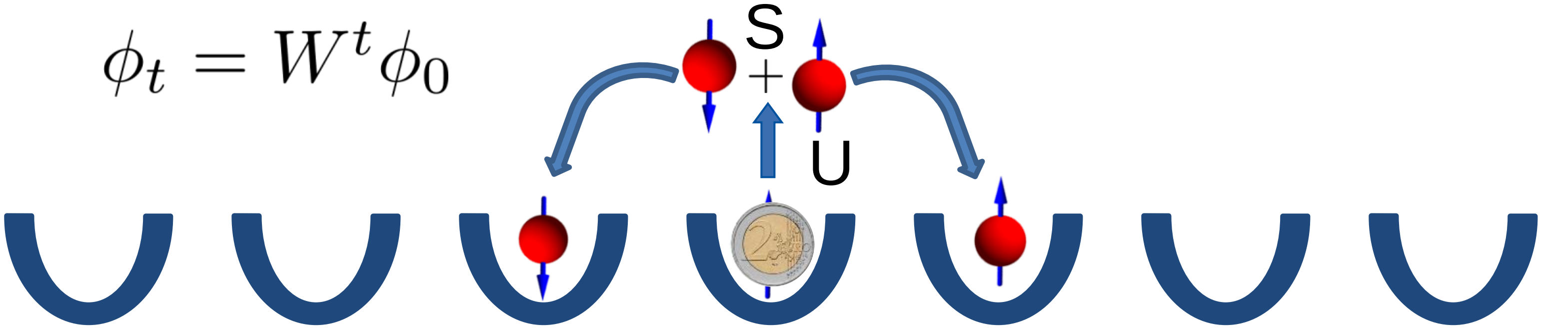

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Example: Hadamard Walk

- Walk operator

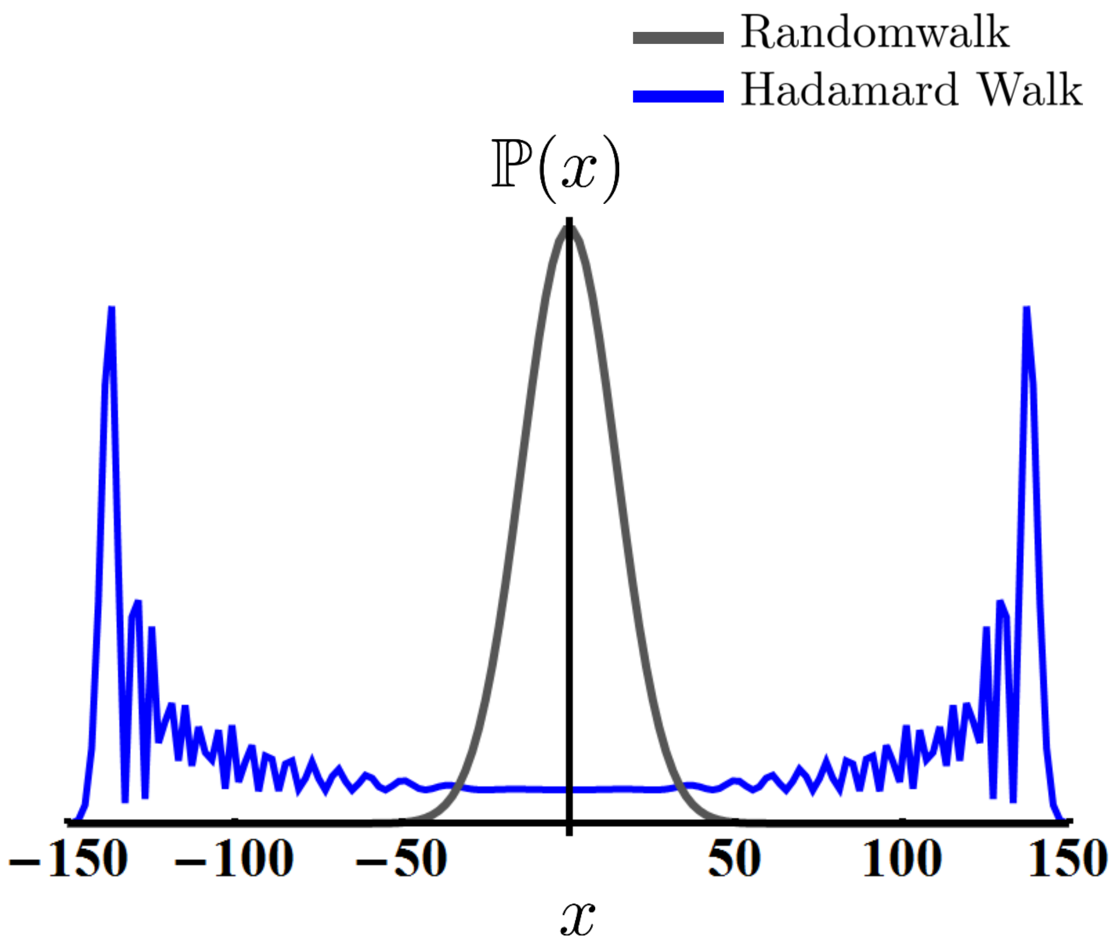
$$W = S\,(\mathbb{1} \otimes U)$$

- Hadamard Coin

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

- Initial state

$$\phi_0 = |0\rangle \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$



200 time steps

Asymptotic position distribution

- Position observable Q_t
- Characteristic function of $\frac{Q_t}{t^\alpha}$

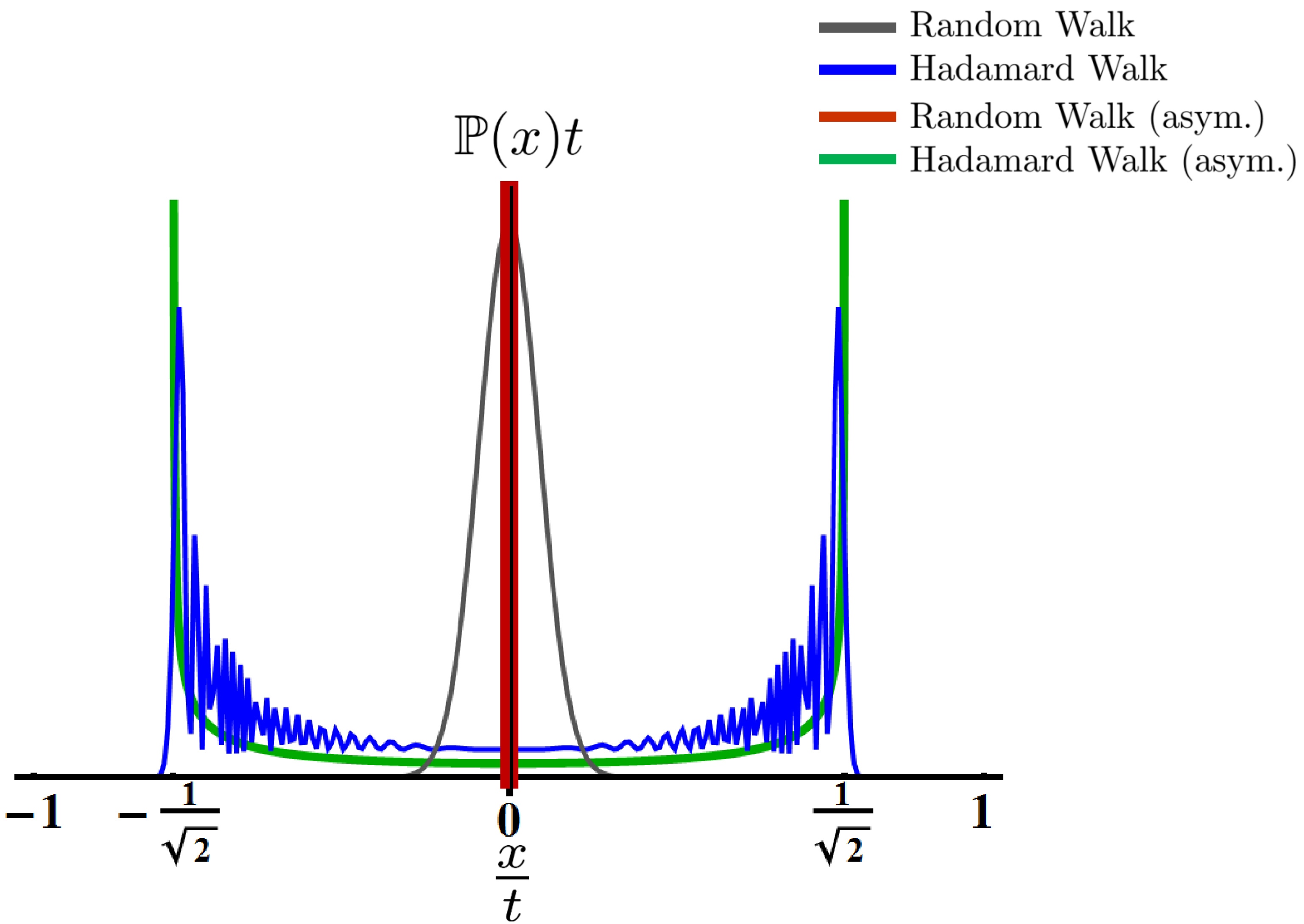
$$\langle e^{i\lambda \frac{Q_t}{t^\alpha}} \rangle = \mathrm{tr}(\rho W^t[e^{i\lambda \frac{Q}{t^\alpha}}])$$

- Find minimal α for the existence of

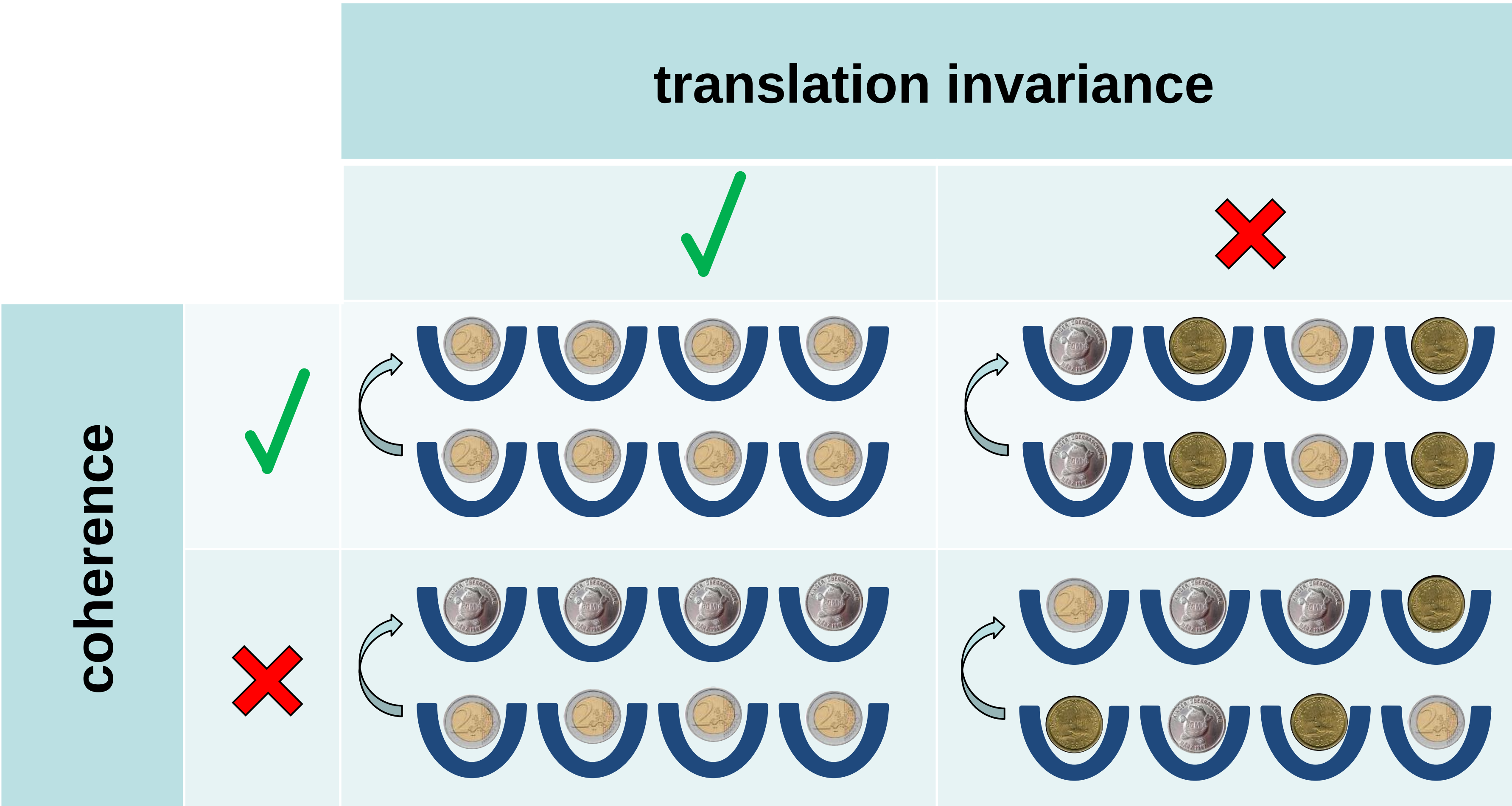
$$\lim_{t\to\infty} \langle e^{i\lambda \frac{Q_t}{t^\alpha}} \rangle$$

Ballistic scaling	**Diffusive scaling**
$\frac{Q_t}{t}$	$\frac{Q_t}{\sqrt{t}}$

1D Example: Hadamard Walk



Propagation properties



Propagation properties

$\frac{Q_t}{t^\alpha}$		translation invariance	
		✓	✗
coherence	✓	Ballistic transport $\sim t$	Anderson localisation ~ 1
coherence	✗	Diffusive transport $\sim \sqrt{t}$	Diffusive transport $\sim \sqrt{t}$

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Interacting Quantum Walks

- Two particles on the line
- Free evolution: $W \otimes W$
- Projection on collision space: $P_{coll} = \sum_x |x, x\rangle\langle x, x| \otimes \mathbb{1}_C$
- Interaction on collision:

$$W_I = W \otimes W((\mathbb{1} - P_{coll}) + C \cdot P_{coll})$$

- Coin on collision: $C \in SU(2)$

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Example: Interacting Hadamard Walk

- Two particles on the line
- Free evolution: $W \otimes W = S \otimes S(\mathbb{1} \otimes H) \otimes (\mathbb{1} \otimes H)$
- Projector on collision space: $P_{coll} = \sum_x |x, x\rangle\langle x, x| \otimes \mathbb{1}_C$
- Interaction on collision:

$$W_I = W \otimes W((\mathbb{1} - P_{coll}) + e^{i\gamma} P_{coll})$$

- Interaction phase: γ
- Initial state: $|0\rangle \otimes 1/\sqrt{2}(|00\rangle - |11\rangle)$
- Walk preserves symmetric/antisymmetric subspaces

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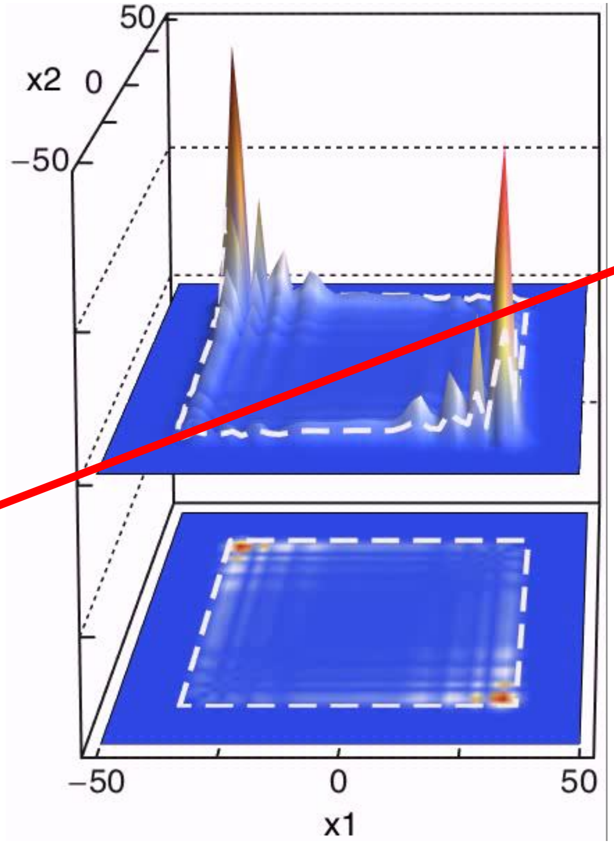
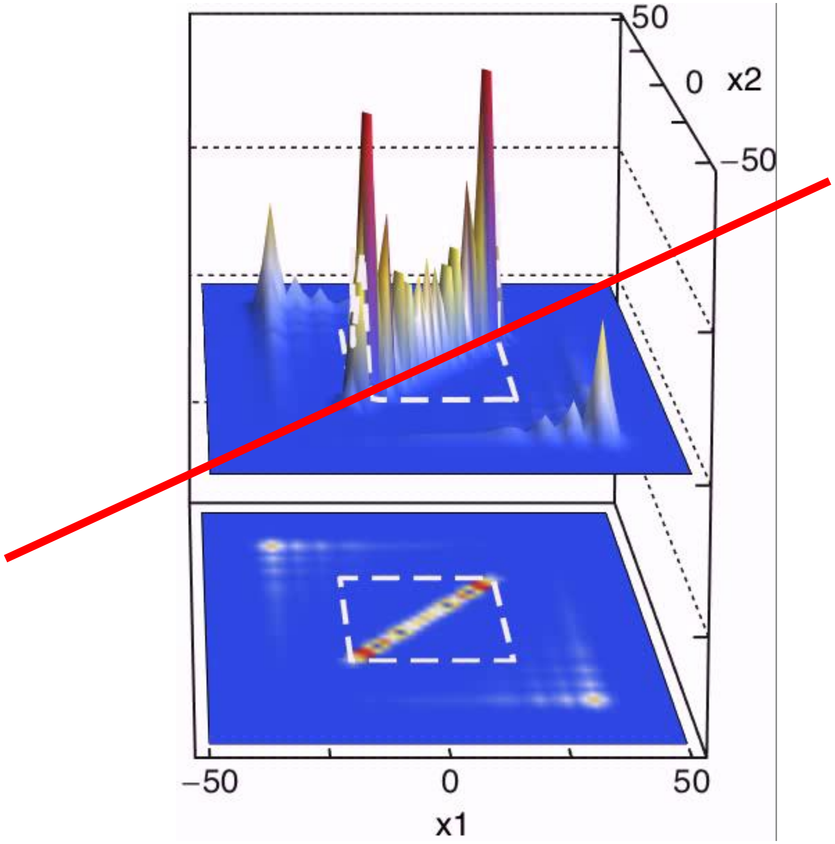
Interacting Hadamard Walk

$$\phi_0 = |0\rangle \otimes \frac{1}{\sqrt{2}} \left(|00\rangle - |11\rangle\right)$$

Free evolution **Interaction**





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Fourier Description I

- Walk operator

$$W_I = W \otimes W \left(\mathbb{1} + (C-1) P_{coll} \right)$$

- Free evolution

$$W \otimes W(p_1, p_2) = W(p_1) \otimes W(p_2) = S(p_1) \otimes S(p_2) \cdot U \otimes U$$

$$\left(W \otimes W \psi \right) (p_1, p_2) = W(p_1) \otimes W(p_2) \cdot \psi(p_1, p_2)$$

$$S(p_1) \otimes S(p_2) = \begin{pmatrix} e^{i(p_1+p_2)} & 0 & 0 & 0 \\ 0 & e^{i(p_1-p_2)} & 0 & 0 \\ 0 & 0 & e^{-i(p_1-p_2)} & 0 \\ 0 & 0 & 0 & e^{-i(p_1+p_2)} \end{pmatrix}$$

Fourier Description II

- Write Walk operator in terms of $p = p1 + p2$ and $k = (p1 - p2)/2$

$$W_I(p,k) = W \otimes W(p,k)\left(\mathbb{1} + (C-1)P_{coll}\right)$$

$$P_{coll} = \sum_x |x,x\rangle\langle x,x| \otimes \mathbb{1}_C$$

$$P_{coll}\psi(p,k) = \int \mathrm{d}k\ \psi(p,k)$$

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Conserved by translation invariance. External parameter

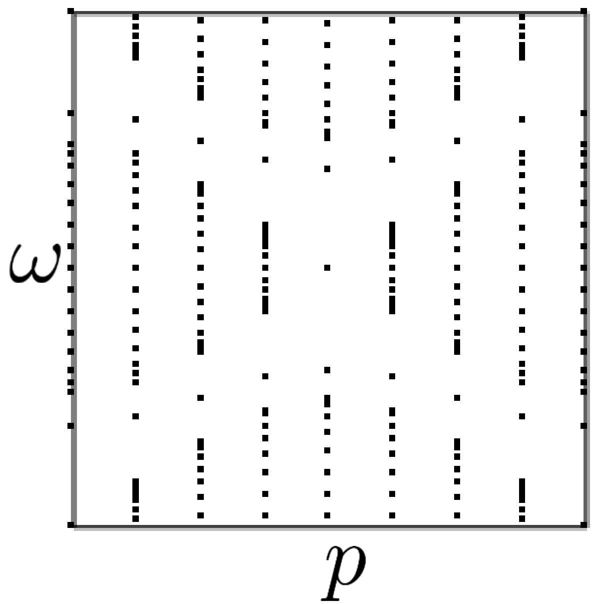
Walk in this variable is perturbed by P_{coll} on subspace of k - constant functions

$\Longleftrightarrow$

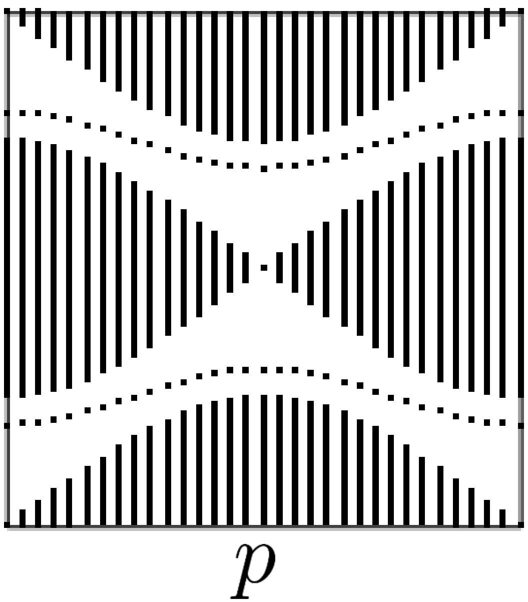
Family of 1D QWs with perturbation at the origin indexed by p

Interacting Hadamard Walk

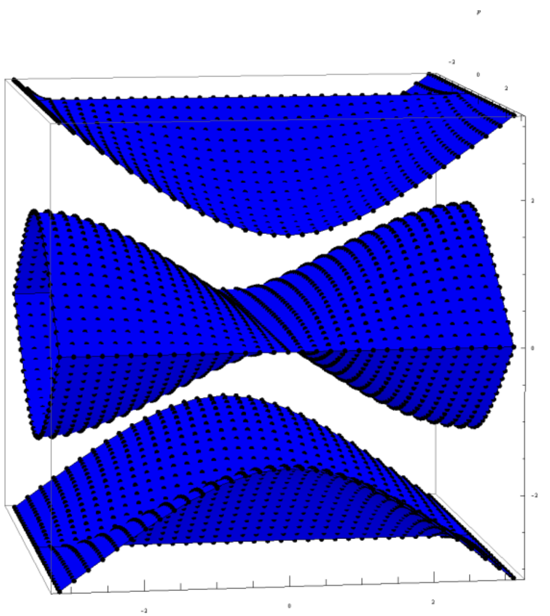
Jointly diagonalize $p = p_1 + p_2$ and ω on a ring of length L



L=8



L=32



Compare with free band structure depth=p_1-p_2

Quantum Walk with Point Perturbation

$$W = W_0(\mathbb{1} + (C-1)P_{x=0})$$

- $P_{x=0}$ projection onto the subspace $x = 0$
- Finite rank perturbation $\Rightarrow$ essential spectrum unchanged
- Look for eigenvalues

$$(W\psi)(k) = z\psi(k)$$

- $P_{x=0}\psi(k) = \Psi$ independent of k

$$(W_0 - z)\psi(k) = W_0(k)(C-1)P_{x=0}\psi(k)$$

- Look for eigenvalues in band gap of W_0

$$\psi(k) = (W_0 - z)^{-1}W_0(k)(C-1)\Psi$$

- Consistency condition: $P_{x=0}\psi(k) = \Psi$

$$\Psi = \int \mathrm{d}k \left((W_0(k) - z)^{-1}W_0(k)\right)(C-1)\Psi$$

Interacting Quantum Walks

$$\Psi_p = \int \mathrm{d}k \left((W_p(k) - z_p)^{-1} W_p(k) \right) (C - 1)\Psi_p$$
$$= R_p(z_p)$$

$$C\Psi_p = (\mathbb{1} - R_p(z_p)^{-1})\Psi_p$$

Result: For all values z_p in the band gap, there is an interaction C such that z_p is an eigenvalue of $W_p(k)$. The corresponding eigenvectors $\Psi_p(x_1 - x_2)$ satisfy

$$\|\Psi_p(x)\| \leq \frac{C_1}{\mathrm{dist}(\sigma(W_p), z_p)} e^{-\mathrm{dist}(\sigma(W_p), z_p) C_2 \|x\|}$$

Example: Interacting Hadamard Walk

- Two particles on the line
- Free evolution: $W \otimes W = (S \otimes S)\left((\mathbb{1} \otimes H) \otimes (\mathbb{1} \otimes H)\right)$
- Projection on collision space: $P_{coll} = \sum_x |x, x\rangle\langle x, x| \otimes \mathbb{1}_C$
- Interaction on collision:

$$W_I = W \otimes W\left((\mathbb{1} - P_{coll}) + e^{i\gamma} P_{coll}\right)$$

- Interaction phase: γ
- Initial state: $|0\rangle \otimes 1/\sqrt{2}(|00\rangle - |11\rangle)$
- Walk preserves symmetric/antisymmetric subspaces

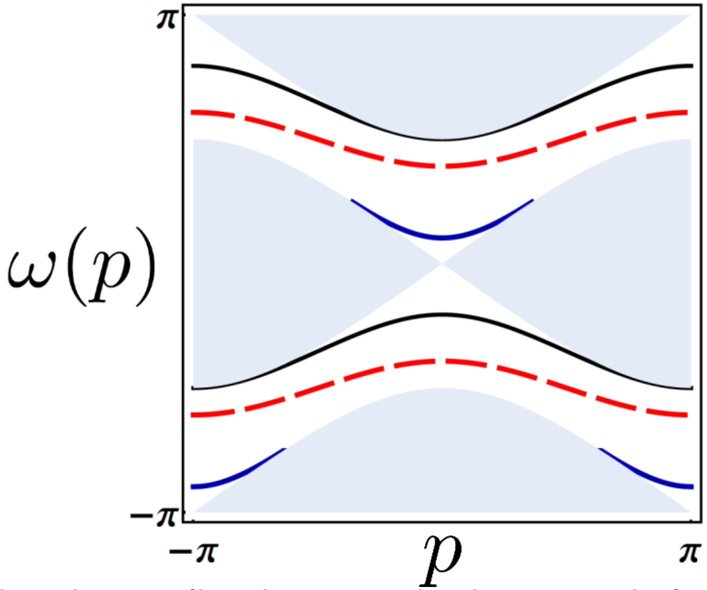
Interacting Hadamard Walk

Result: $\Psi = 1/\sqrt{2}(|00\rangle - |11\rangle)$

- Explicit formula for quasi-energy of the bound state.

$$e^{i\omega} = \frac{e^{i\gamma}}{2e^{i\gamma}-1}\left(\cos p \pm i\sqrt{\sin^2 p + 4(1-\cos\gamma)}\ \right)$$

$$\sin(\omega)\sin(\gamma-\omega) > 0$$

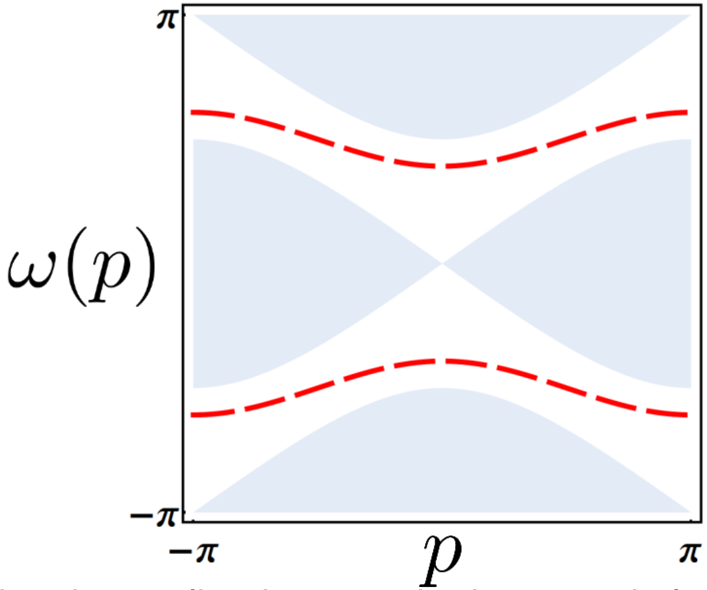


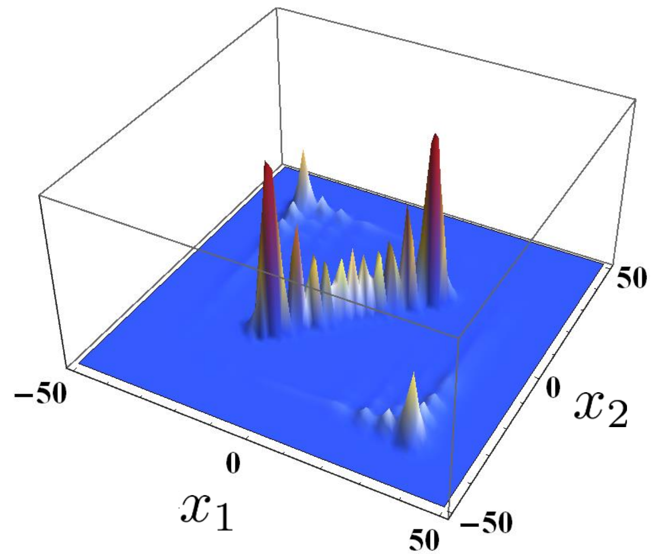
A.Ahlbrecht, A. Alberti, D.Meschede, V.B.Scholz, AHW, R.F. Werner New J. Phys. 14 (2012)
Y.Lahini, M.Verbin, S.D.Huber, Y.Bromberg, R.Pugatch, Y.Silberberg; Phys. Rev. A 86, (2012)
A.Schreiber, A.Gábris, P.Rohde, K.Laiho, M.Štefaňák ,V.Potoček, C.Hamilton ,I.Jex, C.Silberhorn Science (2012)

Interacting Hadamard Walk

Result: $\Psi = 1/\sqrt{2}(|00\rangle - |11\rangle)$

- Explicit formula for quasi-energy of the bound state.
- Effective theory of molecule as QW





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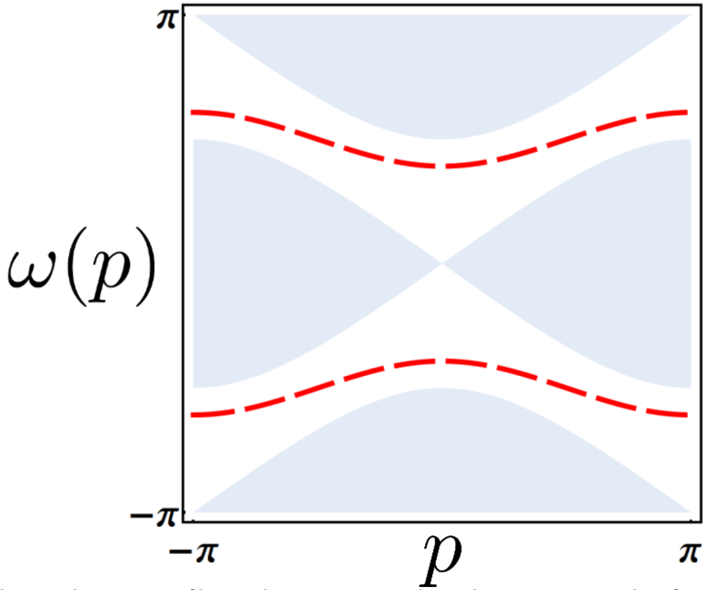

$$W_{mol} = S \cdot C_\gamma$$

$$C_\gamma = \frac{1}{2e^{i\gamma}-1} \begin{pmatrix} e^{i\gamma} & \sqrt{2}(e^{i\gamma}-1) \\ \sqrt{2}e^{i\gamma}-1) & e^{i\gamma} \end{pmatrix}$$

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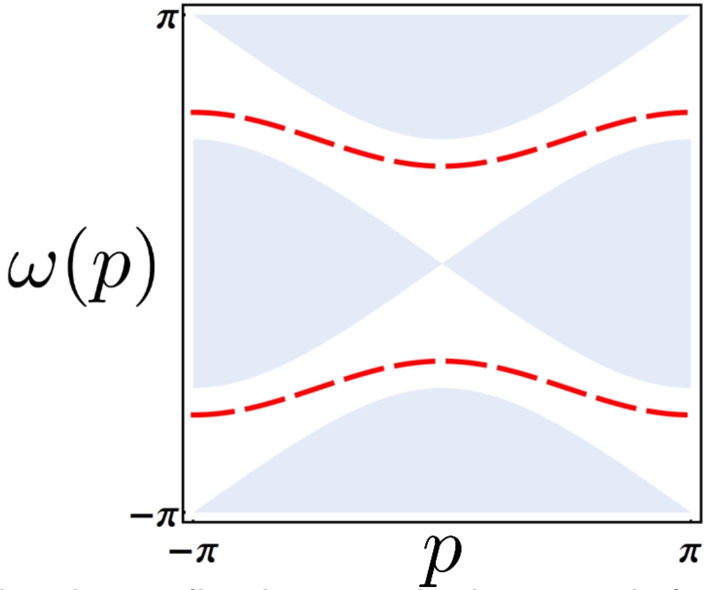
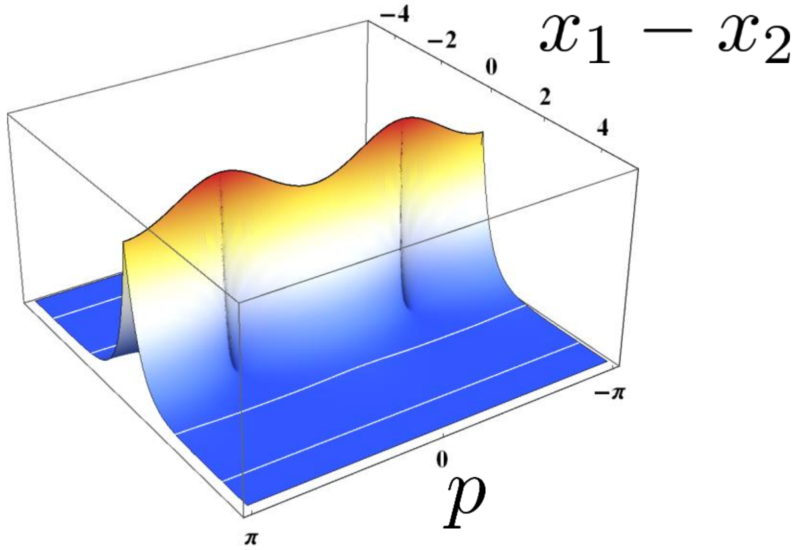
Interacting Hadamard Walk

Result: $\Psi = 1/\sqrt{2}(|00\rangle - |11\rangle)$

- Explicit formula for quasi-energy of the bound state.
- Effective theory of molecule as QW
- Molecule exponentially localized





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Outline

- Propagation properties
- Bound states in interacting quantum walks
- Recurrence properties

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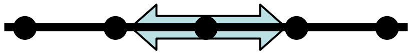
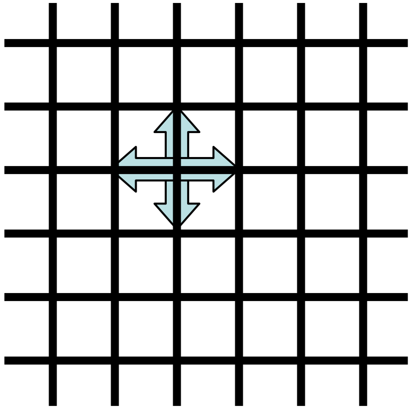
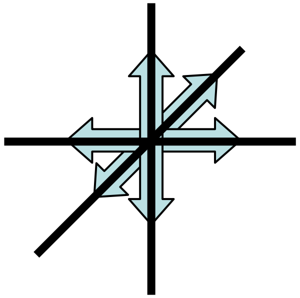
- Propagation properties
- Bound states in interacting quantum walks
- **Recurrence properties**

Recurrence in Random Walks

George Pólya







Does the walker return with certainty?

„Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz“

$d \leq 2$	$d > 2$
recurrent	transient

Georg Pólya; Mathematische Annalen 84(2), (1921)

Markov Process

- Countable state space X
- Transition matrix (P_{xy})
- Probability to move from x to y P_{xy}
- Trajectory $(x, x_1, x_2, x_3, x_4, \ldots, x_{n-1}, y)$

$$\mathbb{P}\left(x, x_1, , x_2, \ldots, x_{n-1}, y\right) = P_{xx_1} \cdot P_{x_1 x_2} \cdots P_{x_{n-1} y}$$

- Fix initial state $0 \in X$
- Probability to return to 0 in exactly n steps

$$p_n := \sum_{x_1, \ldots, x_{n-1}} P_{0x_1} \cdot P_{x_1 x_2} \cdots P_{x_{n-1} 0} = P^n_{00}$$

Georg Pólya; Mathematische Annalen 84(2), (1921)

Return Probabilities

- Return in exactly n steps:

$$p_n = \sum_{x_1,\ldots,x_{n-1}} P_{0x_1} \cdots P_{x_{n-1}0}$$

- First return after exactly n steps (conditioned)

$$q_n = \sum_{\substack{x_1,\ldots,x_{n-1} \\ x_i \neq 0}} P_{0x_1} \cdots P_{x_{n-1}0}$$

Generating functions

$$\widehat{p}(z) = \sum_n z^n p_n \qquad \widehat{q}(z) = \sum_n z^n q_n$$

Recurrence: $R = \sum_n q_n = 1$

Renewal Equation

$$p_n = \sum_{x_1,\ldots,x_{n-1}} P_{0x_1} \cdot P_{x_1 x_2} \cdots P_{x_{n-1}0} = \sum_{k=0}^{n} p_k q_{n-k}$$

Renewal equation: $\widehat{q}(z) = 1 - \frac{1}{\widehat{p}(z)}$

Recurrence criterium:

$$R = 1 = \widehat{q}(1) = 1 - \frac{1}{\widehat{p}(1)}$$

Polya criterium:

$$\text{divergence of } \widehat{p}(1) = \sum_n p_n$$

Recurrence in Time Discrete Quantum Systems

Scenario: separable Hilbert space $\mathcal{H}$

unitary operator $W \in \mathcal{B}(\mathcal{H})$

evolution: $W^t\phi$, $t \in \mathbb{N}$

Question: Given $\phi \in \mathcal{H}$, does the system return with certainty to this initial state?

Return Amplitudes

- Return after exactly n steps

$$\mu_n = \langle \phi | W^n \phi \rangle$$

- Generating function

$$\widehat{\mu}(z) = \sum_n \mu_n z^n = \langle \phi | \sum_n z^n W^n \phi \rangle$$

Conceptional problem: First return probabilities n .

Idea: Use renewal equation

$$\widehat{q}(z) = 1 - \frac{1}{\widehat{p}(z)}$$

Return Amplitudes

- Return after exactly n steps

$$\mu_n = \langle \phi | W^n \phi \rangle$$

- Generating function

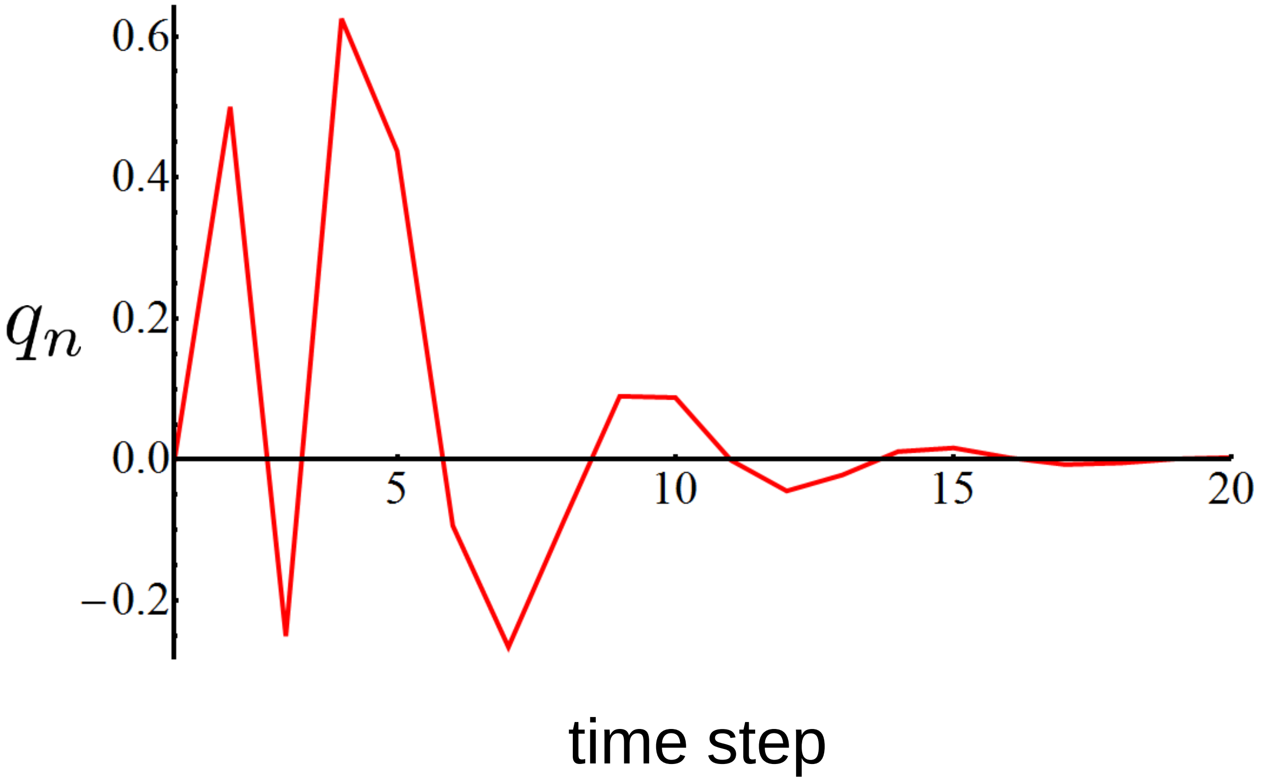
$$\widehat{\mu}(z) = \sum_n \mu_n z^n = \langle \phi | \sum_n z^n W^n \phi \rangle$$

Conceptional problem: First return probabilities n .

Idea: Us… ewal equation

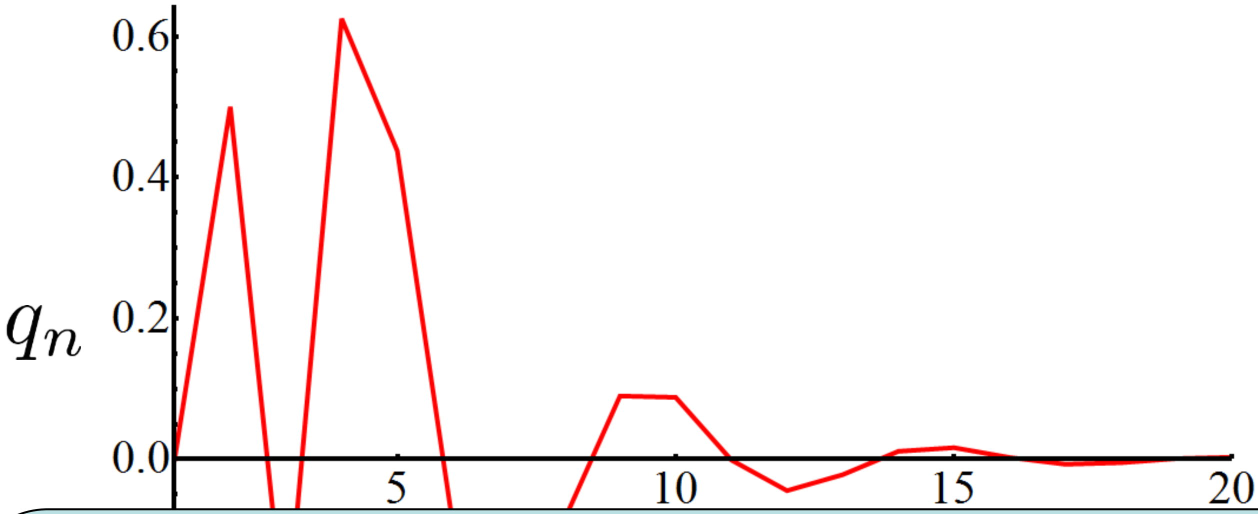
$\widehat{a}(z) = 1 - \ldots$

Simple Counter Example



$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \phi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Simple Counter Example



Way out:
Directly use classical Polya criterium for $|\mu_n|^2$

$$\widehat{p}(1) = \sum_n |\mu_n|^2 = \infty$$

M. Stefanak, I. Jex, T. Kiss Phys. Rev. Lett. 100, 020501 (2008)

Operational Approach

Test for return in each time step

- projective measurement: $\{|\phi\rangle\langle\phi|,\ (\mathbb{1} - |\phi\rangle\langle\phi|)\}$
- **Modified dynamics:**
  1. Unitary time step $\phi_n \mapsto W\phi_n$
  2. Measurement: System in state ϕ ?

Yes	No
System returned. End of experiment.	System in state $(\mathbb{1} - |\phi\rangle\langle\phi|)W\phi_n$

$$\widetilde{W} = (\mathbb{1} - |\phi\rangle\langle\phi|)W$$

First Return Amplitudes

- First return after n steps

$$a_1 = \langle \phi | \, W \; \phi \rangle \qquad a_n = \langle \phi | \, W \widetilde{W}^{n-1} \; \phi \rangle$$

- Generating function

$$\widehat{a}(z) = \sum_n a_n z^n$$

- Total return probability

$$R = \sum_n |a_n|^2 = 1 - \lim_{t \to \infty} ||\widetilde{W}^t \phi||^2$$

Definition: (ϕ, W) recurrent iff $R = 1$.

Renewal Equation

- Generating function

$$\widehat{a}(z) = \sum_{n=1} a_n z^n = z\langle\phi|\, W \left(\mathbb{1} - z\widetilde{W}\right)^{-1} \phi\rangle$$

- W and $\widetilde{W}$ differ by rank 1 perturbation: Krein Formula

$$z\langle\phi|\, W \left(\mathbb{1} - z\widetilde{W}\right)^{-1} \phi\rangle = 1 - \left(\langle\phi|\, (\mathbb{1} - zW)^{-1} \phi\rangle\right)^{-1}$$

- Identify scalar product on RHS with $\widehat{\mu}(z)$

Renewal-equation

$$\widehat{a}(z) = 1 - \frac{1}{\widehat{\mu}(z)}$$

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Random Walk vs. Quantum Case

	Random Walk: probabilities	Quantum Case: amplitudes
Return	p_n	μ_n
First return	q_n	a_n
Return probability	$\sum_n q_n$	$\sum_n \lvert a_n \rvert^2$
Renewal equation	$\widehat{q}(z) = 1 - \frac{1}{\widehat{p}(z)}$	$\widehat{a}(z) = 1 - \frac{1}{\widehat{\mu}(z)}$

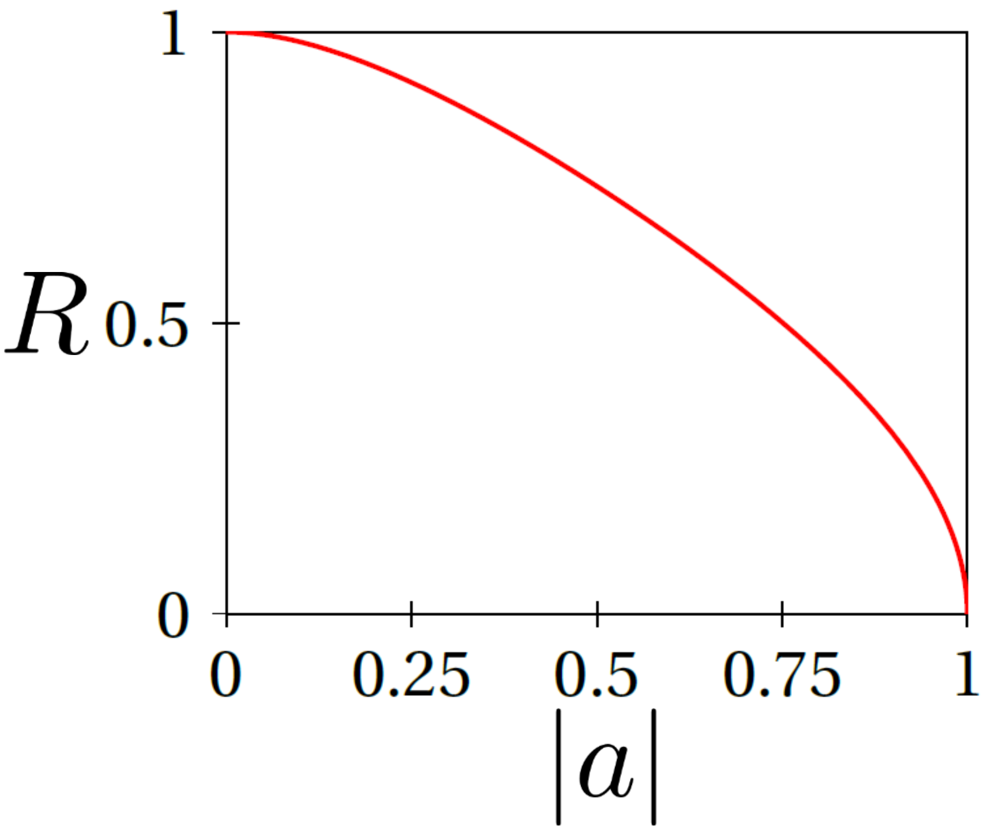
F.A. Grünbaum, L. Velázquez, A. H. Werner, R. F. Werner; Com. Math. Phys. 320(2) (2013)

Return Probability of 1D Quantum Walks

$$W = S \cdot (\mathbb{1} \otimes U)$$

$$U = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$$

$$\phi = |x, \pm\rangle$$



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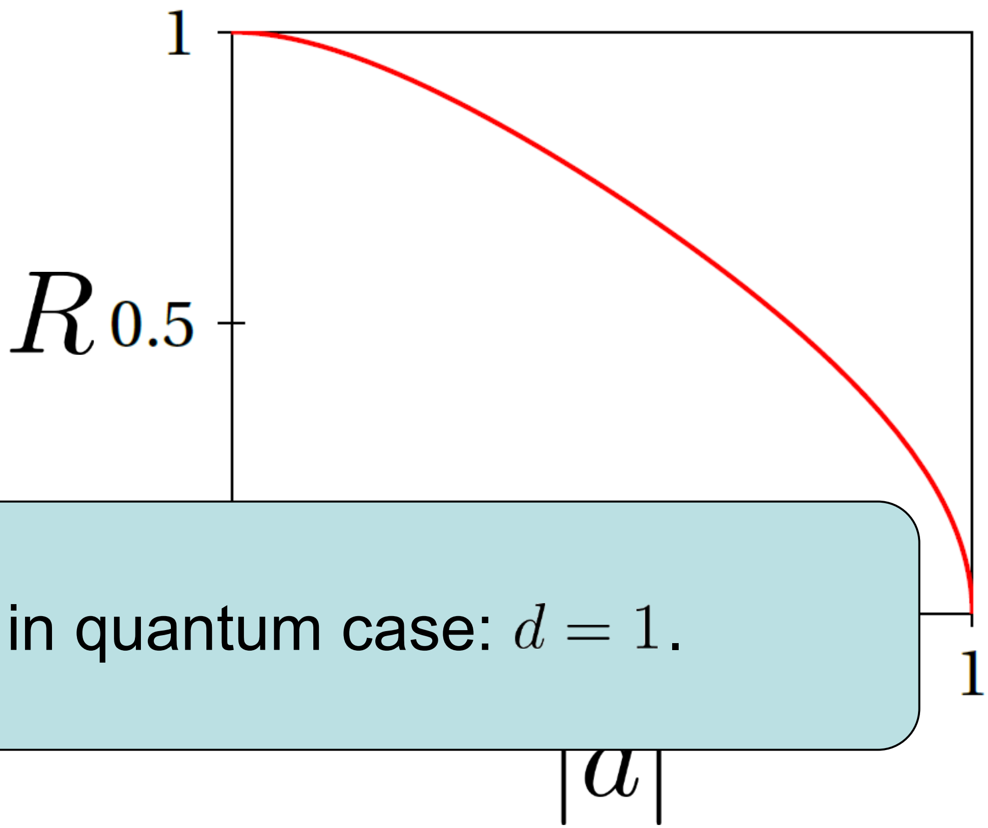

Critical dimension in quantum case: $d = 1$.

Recurrence Criteria

Characterization in terms of spectral measure μ_ϕ

$$\langle \phi | f(W)\ \phi \rangle = \int_{-\pi}^{\pi} f(e^{i\theta}) \mu_\phi(\mathrm{d}\theta)$$

For matrices

$$\langle \phi | f(W)\ \phi \rangle = \sum_k f(e^{i\theta_k})\ |\langle \phi | k \rangle|^2$$

Recurrence Criteria

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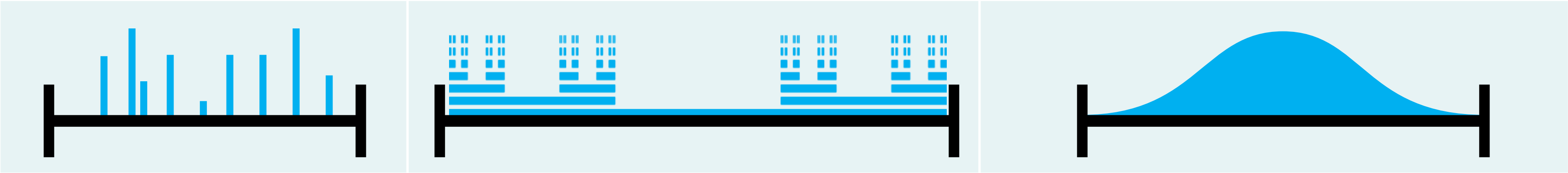
For matrices

$$\langle\phi| f(W)\ \phi\rangle = \sum_k f(e^{i\theta_k})\ |\langle\phi|k\rangle|^2$$

$$\mathcal{H} = H_{pp} \oplus H_{sc} \oplus H_{ac}$$

pure point	**singular continuous**	**absolutely continuous**



Recurrence Criteria

Characterization in terms of spectral measure μ_ϕ

$$\langle \phi | f(W)\ \phi \rangle = \int_{-\pi}^{\pi} f(e^{i\theta}) \mu_\phi(\mathrm{d}\theta)$$

Theorem: (ϕ, W) is recurrent iff μ_ϕ has no absolutely continuous component.

$$\mathcal{H} = H_{pp} \oplus H_{sc} \oplus H_{ac}$$

pure point	**singular continuous**	**absolutely continuous**

F.A. Grünbaum, L. Velázquez, A. H. Werner, R. F. Werner; Com. Math. Phys. 320(2) (2013)

RAGE Theorem

(K_n) sequence of compact operators strongly convergent to the identity, U unitary operator

$$\mathcal{H}_c = \{\phi \in \mathcal{H} \; ; \; \lim_{n\to\infty} \lim_{T\to\infty} \frac{1}{T+1} \sum_{t=0}^{T} \|K_n U^t \phi\| = 0\}$$

$$\mathcal{H}_{\rm pp} = \{\phi \in \mathcal{H} \; ; \; \lim_{n\to\infty} \sup_{t\geq 0} \| (\mathbb{1} - K_n) U^t \phi\|^2 = 0\}$$

$$\mathcal{H} = H_{pp} \oplus H_{sc} \oplus H_{ac}$$

pure point | **singular continuous** | **absolutely continuous**

Comparison

Recurrence

- (W, ϕ) is recurrent iff μ_ϕ contains no absolutely continuous component

- Distinguishes between singular and non-singular spectrum

RAGE theorem

- ϕ is localized iff the spectral measure μ_ϕ is pure point

- Distinguishes between continuous and pure point spectrum

Proof idea I: Measures on the unit circle

- Given a probability measure μ on S^1, the unit circle, define for $z \in \mathbb{C}$ with $|z| < 1$ two analytic functions
  - Stieltjes function: $S(z) = \int \frac{\mu(\mathrm{d}u)}{1-uz}$
  - Schur function : $f(z) = \frac{1}{z}\frac{\overline{S}(z)-1}{\overline{S}(z)}$
- Boundary behaviour of S and f for $|z| \to 1$ characterizes μ.

Theorem: The absolutely continuous part of μ is supported on the subset of S^1, where $|f(z)| < 1$.

Proof idea II

- Identify RHS of renewal equation with Schur function

$$\widehat{a}(z) = 1 - \frac{1}{\widehat{\mu}(z)} = z\overline{f}(z)$$

- For (ϕ, W) to be recurrent we need

$$1 = \sum_t |a_t|^2 = \lim_{r\to 1} \sum_t r^{2t} |a_t|^2$$

- Using $\widehat{a}(t) = z\overline{f}(z)$ this implies for the Schur function

$$\lim_{r\to 1} \int_0^{2\pi} \mathrm{d}t |f(re^{it})|^2 = 1$$

- Since $|f(z)|$ bounded by 1 we need

$$\lim_{r\to 1} |f(re^{it})| = 1$$

for almost all $0 \leq t \leq 2\pi$, which is equivalent to μ_ϕ having no absolutely continuous part.

Expected Return Time

- Given first return amplitudes

$$a_t = \langle\phi|W\widetilde{W}^{t-1}|\phi\rangle$$

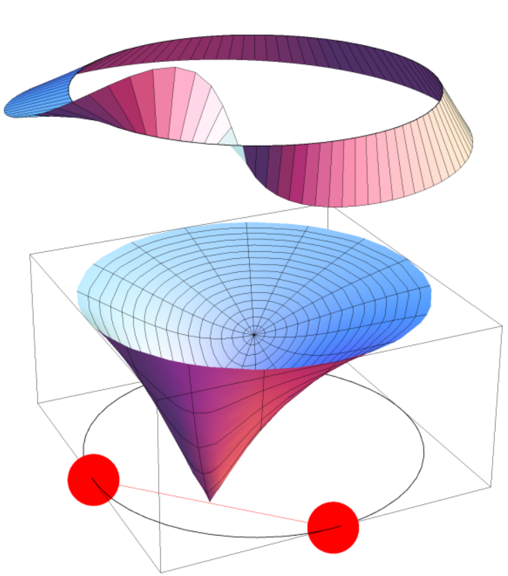
- Consider expected return time

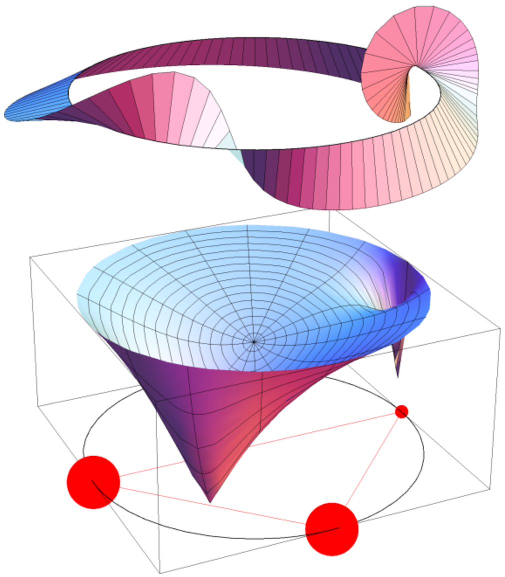
$$\tau = \sum_t |a_t|^2 t$$

Result: If the pair (ϕ, W) is recurrent, the expected return time τ is infinite or an integer!
τ counts point masses in μ_ϕ.

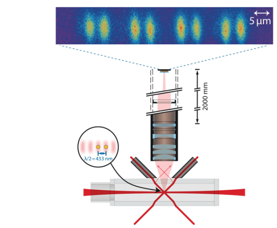
- Proof idea: Identify τ with winding number of the phase of the Schur function on the unit circle

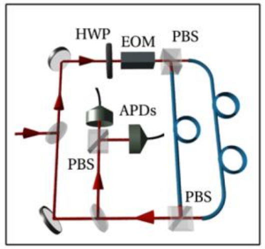
Expected return time

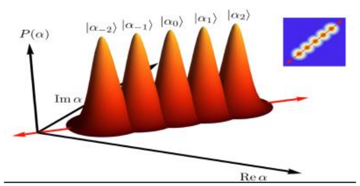


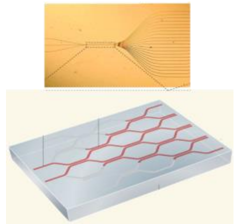


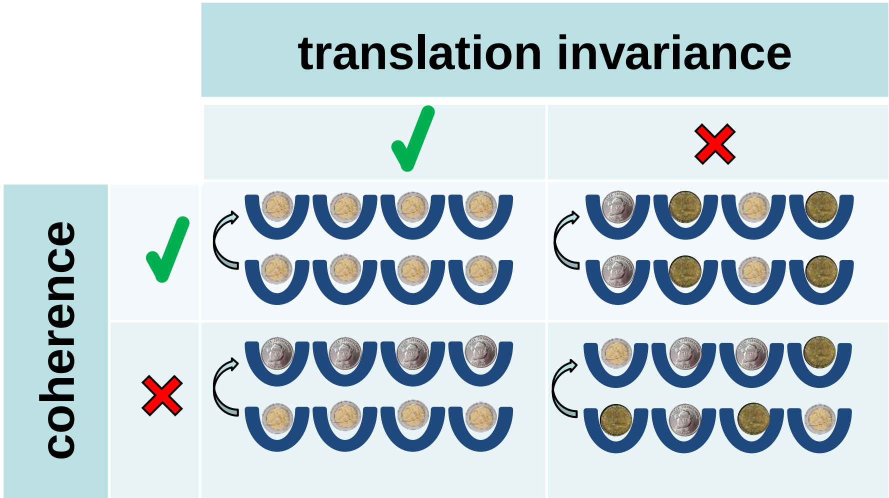
Summary

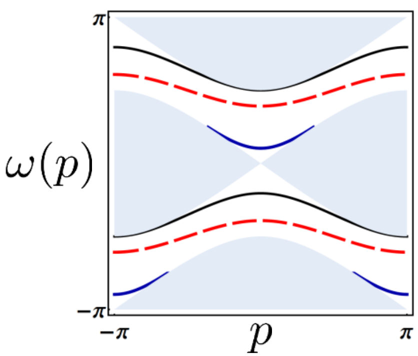


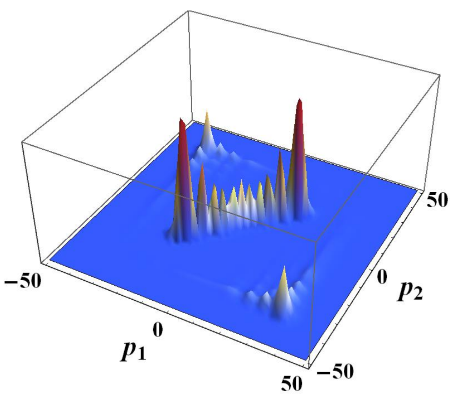


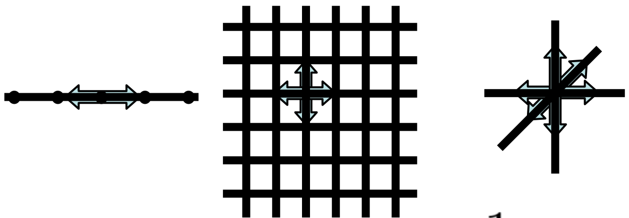




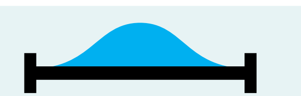








$$\widehat{a}(z) = 1 - \frac{1}{\widehat{\mu}(z)}$$

Pure Point spectrum	Singular continuous spectrum	absolutely continuous spectrum
		

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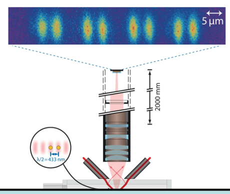
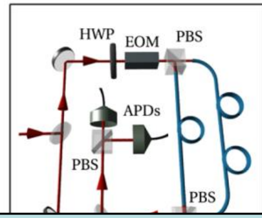
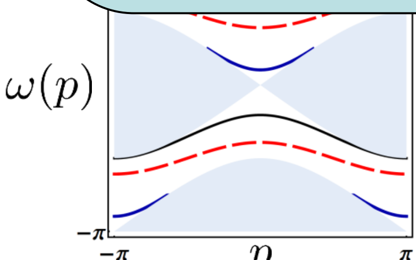
Summary

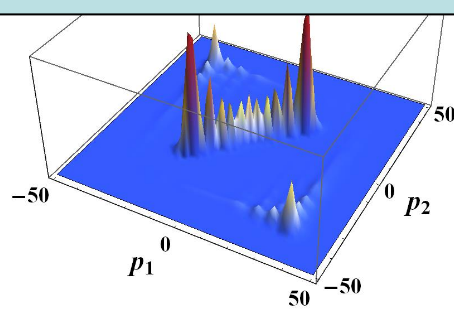




translation invariance

Thank you for your attention!





$$\widehat{a}(z) = 1 - \frac{1}{\widehat{\mu}(z)}$$

Pure Point spectrum	Singular continuous spectrum	absolutely continuous spectrum

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References I

- A. Ahlbrecht, V.B. Scholz, A. H. Werner; J. Math. Phys. 52, 102201 (2011)
- A. Ahlbrecht, H. Vogts, AHW, and R. F. Werner J. Math. Phys. 52, 042201 (2011)
- G. Grimmett, S. Janson, P.F. Scudo; Phys. Rev. E, 69, 026119 (2004).
- F.A. Grünbaum, L. Velázquez, A. H. Werner, R. F. Werner; Com. Math. Phys. 320(2) (2013)
- A. Joye; CMP 307(1) (2011)
- A. Joye, M. Merkli; J. Stat. Phys. 140(6) (2010)
- M. Karski, L. Förster, JM. Choi, A. Steffen, W. Alt, D. Meschede, A. Widera; Science 325 (2009)
- N. Konno, J. Math. Soc. Japan Volume 57, Number 4 (2005)
- R. Matjeschk, A. Ahlbrecht, M. Enderlein, Ch. Cedzich, A. H. Werner, M. Keyl, T. Schaetz, R. F. Werner; Phys. Rev. Lett. 109, 240503 (2012)
- A. Peruzzo, M. Lobino, J. C. F. Matthews, N. Matsuda, A. Politi, K. Poulios, X. Zhou, Y. Lahini, N. Ismail, K. Wörhoff, Y. Bromberg, Y. Silberberg, M. G. Thompson, J. L. O'Brien; Science, 329(5998) (2010)
- G. Pólya; Mathematische Annalen 84(2), (1921)

References II

- A. Schreiber, K. N. Cassemiro, V. Potoček, A. Gábris, I. Jex, C. Silberhorn Phys. Rev. Lett. 106, 180403 (2011)
- H. Schmitz, R. Matjeschk, Ch. Schneider, J. Glueckert, M. Enderlein, T. Huber, and T. Schaetz; Phys. Rev. Lett. 103, 090504 (2009)
- ucm.es: http://pendientedemigracion.ucm.es/info/giccucm/index.php/Quantum_Computation.html Zugriff: 03.07.13
- wikipedia.org: http://en.wikipedia.org/wiki/D-Wave_Systems Zugriff. 03.07.13
- iap.uni-bonn.de: http://quantum-technologies.iap.uni-bonn.de/ Zugriff. 03.07.13
- F. Zähringer, G. Kirchmair,, R. Gerritsma, E. Solano, R. Blatt and C. F. Roos; Phys. Rev. Lett. 104, 100503 (2010)