Full statistics of erasure processes: Isothermal adiabatic theory and a statistical Landauer principle

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Solid Math
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Taming Maxwell's demon: a never ending story made short

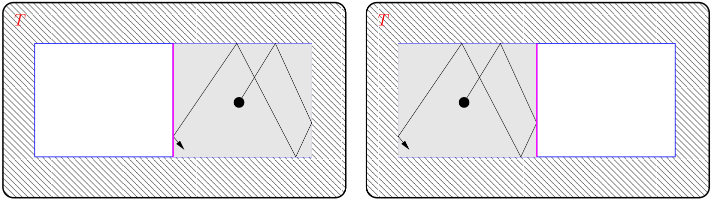
- 1871: Maxwell's demon violates the 2$^{\text{nd}}$ Law
- 1929: Szilard's engine converts information into work
- 1956: Brillouin: irreversibility of quantum measurement processes
- 1961: Landauer: *logically irreversible* operations dissipate heat

$$\Delta Q = k_{\mathrm{B}} T \log 2 \text{ per bit}$$

- 1982: Bennett exorcises the demon
- 1999: Earman-Norton criticism...
- ... many attempts to “prove” Landauer's principle from ”first principles” (stat. mech.) or conceive classical and quantum systems that violate it ...

Thermodynamic "derivation" of Landauer's Principle

The ideal gas 1-bit memory ($pV = k_B T$)



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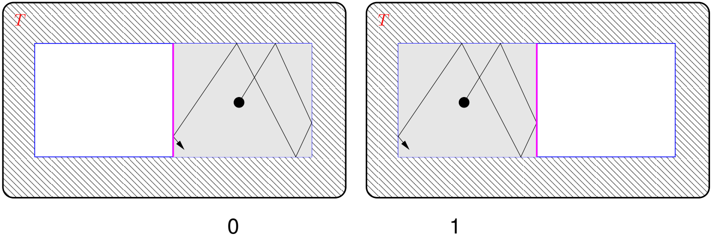
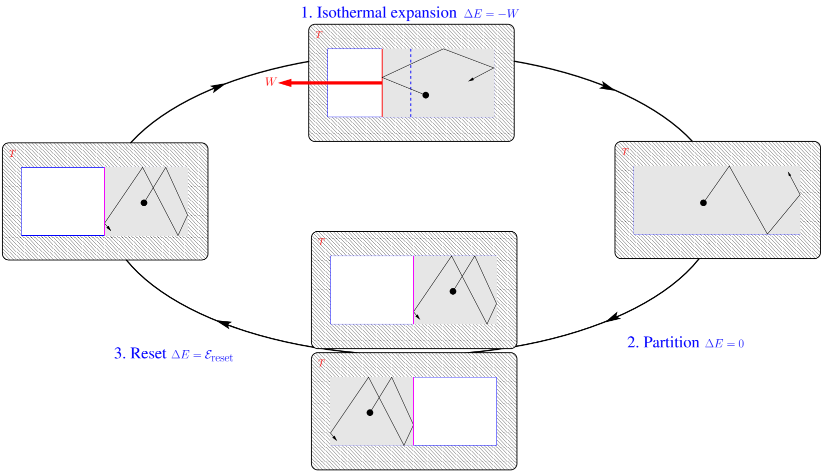


Assume there is a process which perform the reset operation **(0 or 1)** $\rightarrow$ 0 with energy cost

$$\mathcal{E}_{\text{reset}}$$

Thermodynamic “derivation” of Landauer’s Principle

Build a cyclic process – Szilard Engine



Thermodynamic "derivation" of Landauer's Principle

Work extracted during isothermal quasi-static expansion

$$W = \int_{V/2}^{V} p\,dV = \int_{V/2}^{V} \frac{k_\mathrm{B} T}{V}\,\mathrm{d}V = k_\mathrm{B} T \log 2$$

The second law imposes

$$\mathcal{E}_\mathrm{reset} \geq k_\mathrm{B} T \log 2$$

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[Landauer ’61] The energy injected in the reset process is released as heat in the reservoir. $k_B T \log 2$ is the minimal energy dissipated by a reset operation. Moreover

$$k_B T \log 2 = T\Delta S$$

ΔS being the decrease in entropy of the system in the reseting process (erasing entropy). Note that Landauer’s bound $\mathcal{E}_{\mathrm{reset}} \geq T\Delta S$ is saturated by the reverse process of quasi-static isothermal compression.

Landauer's Principle from statistical mechanics

[Earman-Norton 1999, Bennett 2003, Leff-Rex 2003, ...] All known derivations of Landauer's Principle assume the validity of one or another form of the 2^{nd} Law.

[Shizume 1995, Piechocinska 2000, ...] Landauer's Principle from classical and quantum microscopic dynamics of specific systems

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This formulation will definitively not close the philosophical discussions about Maxwell's demon and the relation between thermodynamics and information theory, but at least it provides a sound statement with well defined assumptions.

Landauer's Principle in statistical mechanics [Reeb-Wolf '14]

Finite quantum system $\mathcal{S}$ coupled to finite reservoir $\mathcal{R}$ at temperature $T > 0$

- Finite dimensional Hilbert space $\mathcal{H} = \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}}$, reservoir Hamiltonian $H_{\mathcal{R}}$
- Product initial state + thermal reservoir $\omega_i = \rho_i \otimes \nu_i$

$$\nu_i = \mathrm{e}^{-(\beta H_{\mathcal{R}} + \log Z)}, \qquad \beta = \frac{1}{k_{\mathrm{B}} T}, \qquad Z = \mathrm{tr}\left(\mathrm{e}^{-\beta H_{\mathcal{R}}}\right)$$

- Unitary state transformation $U : \omega_i \mapsto \omega_f = U \omega_i U^*$
- Reduced final states

$$\rho_f = \mathrm{tr}_{\mathcal{H}_{\mathcal{R}}}(\omega_f), \qquad \nu_f = \mathrm{tr}_{\mathcal{H}_{\mathcal{S}}}(\omega_f)$$

- Energy dissipated in the reservoir $\mathcal{R}$:

$$\langle \Delta Q \rangle = \mathrm{tr}((\nu_f - \nu_i) H_{\mathcal{R}})$$

- Decrease in entropy of the system $\mathcal{S}$:

$$\Delta S = S(\rho_i) - S(\rho_f)$$

where $S(\rho) = -k_{\mathrm{B}}\, \mathrm{tr}\,(\rho \log \rho)$ is the von Neumann entropy of ρ

Landauer's Principle in statistical mechanics

Landauer's bound (statistical 2$^{\text{nd}}$ Law)[Reeb-Wolf '14, Tasaki '00]

$$\langle \Delta Q \rangle = T(\Delta S + \sigma), \qquad \sigma \geq 0 \tag{1}$$

$\sigma = 0$ iff $\langle \Delta Q \rangle = T\Delta S = 0$, in which case $\nu_f = \nu_i$, ρ_f and ρ_i being unitarily equivalent.

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Remark 1. If $\mathcal{S}$ is a qubit,

$$\rho_i = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}, \qquad \rho_f = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

then the transformation $\rho_i \to \rho_f$ implements the state change **(0 or 1)** $\to 0$ and

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However, this transition can not be induced by a finite reservoir at positive temperature (more later).

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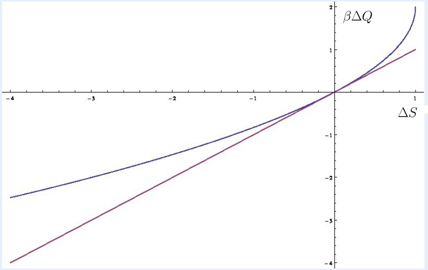
Remark 3. Mathematically: simple, direct consequence of Klein's inequality.

Tightness of Landauer's Bound

[Reeb-Wolf '14] Most of their analysis consists in showing that Landauer's bound is not tight for reservoirs with finite dimensional Hilbert space and deriving tighter bounds in such cases.



Conjecture: *Landauer's Principle can probably be formulated within the general statistical mechanical framework of C^* and W^* dynamical systems and an equality version akin to (1) can possibly be proven.*

Tightness of Landauer's Bound

Macroscopic reservoir should be idealized as infinitely extended
$\rightsquigarrow$ Thermodynamic Limit

Familiar objects (Hamiltonians, density matrices,...) lose their meaning in the Thermodynamic Limit ...

...but other structures emerge (modular theory)

[Jakšić-P'15] Landauer's principle holds for infinitely extended reservoirs, under appropriate and physically reasonable ergodicity hypotheses. Moreover, the bound is saturated by isothermal quasi-static (i.e., adiabatic=infinitely slow) processes. Proof based on the gapless adiabatic theorem [Avron-Elgart'99, Teufel'01, Abou Salem-Fröhlich'05]

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Can we improve Landauer's Principle beyond the expected value $\langle \Delta Q \rangle$?

Improving Landauer's bound: Heat Full Statistics

Strategy: study the balance between ΔQ and ΔS in a quasi–static transition $\rho_i \to \rho_f$ induced by coupling $\mathcal{S}$ to a reservoir $\mathcal{R}$ of finite size L with a time-dependent Hamiltonian

$$H^{(L)}(t/T) = H_{\mathcal{R}}^{(L)} + H_{\mathcal{S}}(t/T) + \lambda(t/T)V$$

in the limits $L, T \to \infty$. Two time-scales:

$$\text{physical time } t \in [0, T], \qquad \text{epoch } s = t/T \in [0, 1]$$

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Practical implementation, assuming $\mathcal{H}_{\mathcal{S}} = \mathbb{C}^d$, $\rho_i = d^{-1}I$ and $\rho_f > 0$:

- $\lambda(0) = 0$: at time $t = 0$, $\mathcal{S}$ and $\mathcal{R}$ being decoupled, in the product state $\omega_i^{(L)} = \rho_i \otimes \nu_i^{(L)}$, measure the total energy of $\mathcal{R} \rightsquigarrow E_i \in \mathrm{sp}(H_{\mathcal{R}}^{(L)})$

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- $H_{\mathcal{S}}(0/1) = -\beta^{-1} \log \rho_{i/f} + F_{i/f}$: for $t \in]0, T[$ the joint system $\mathcal{S} + \mathcal{R}$ evolves according to the time-dependent Hamiltonian $H^{(L)}(t/T)$

$$T^{-1} \mathrm{i}\partial_s U_s^{(L,T)} = H^{(L)}(s) U_s^{(L,T)}$$

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- $\lambda(1) = 0$: at time $t = T$, $\mathcal{S}$ and $\mathcal{R}$ being again decoupled, measure the total energy of $\mathcal{R} \rightsquigarrow E_f \in \mathrm{sp}(H_{\mathcal{R}}^{(L)})$

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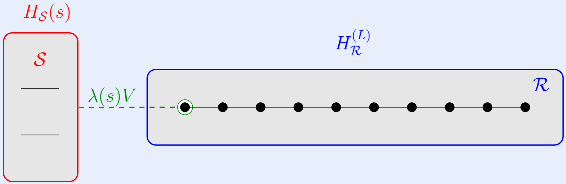
- Full Statistic of dissipated heat = Probability distribution of $\Delta Q = E_f - E_i$

Improving Landauer's bound: Heat Full Statistics

A simple model



Qubit $\mathcal{S}$ coupled to free lattice fermions $\mathcal{R}$

Improving Landauer's bound: Heat Full Statistics

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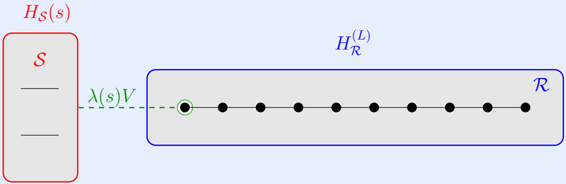

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Quasi-static process reached by taking first $L \to \infty$ and then $T \to \infty$.

Remark. After tracing out $\mathcal{R}$, this regime should be equivalent to the Markovian adiabatic theory [Avron-Fraas-Graf-Grech'12]

Improving Landauer's bound: Heat Full Statistics

Assumption I (Thermodynamic Limit)

For $s \in [0,1]$ and $T > 0$, as $L \to \infty$

$$\mathrm{e}^{\mathrm{i}tH^{(L)}(s)}(\,\cdot\,)\mathrm{e}^{-\mathrm{i}tH^{(L)}(s)} \to \gamma_{(s)}^{t}(\,\cdot\,)$$

$$U_s^{(L,T)*}(\,\cdot\,)U_s^{(L,T)} \to \tau_{(T)}^{s}(\,\cdot\,)$$

$$\eta_s^{(L)}(\,\cdot\,) = \frac{\mathrm{tr}\,\mathrm{e}^{-\beta H^{(L)}(s)}(\,\cdot\,)}{\mathrm{tr}\,\mathrm{e}^{-\beta H^{(L)}(s)}} \to \eta_s(\,\cdot\,)$$

η_s being the unique equilibrium state at temperature β^{-1} for $\gamma_{(s)}^{t}$ (KMS-condition)

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Remark 1. By (I) and the boundary conditions on $\lambda(0/1)$ and $H_{\mathcal{S}}(0/1)$,

$$\eta_0 = \rho_i \otimes \nu_\mathrm{i}, \qquad \eta_1 = \rho_f \otimes \nu_\mathrm{i}$$

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Improving Landauer's bound: Heat Full Statistics

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Remark 2. (II) enforces interaction $\lambda(s)V$ to be non-trivial for $s \in]0,1[$

Improving Landauer's bound: Heat Full Statistics

Applying the gapless adiabatic theorem [Avron-Elgart'99, Teufel'01] in the GNS representation yields the

Isothermal Adiabatic Theorem [Abou Salem-Fröhlich'05, Jakšić-P'14]

If the functions $\lambda(s)$ and $H_{\mathcal{S}}(s)$ are $C^1([0,1])$ and Assumptions (I)-(II) are satisfied then

$$\lim_{T\to\infty} \sup_{s\in[0,1]} \|\eta_0 \circ \tau_{(T)}^s - \eta_s\| = 0$$

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By the previous remark,

$$\lim_{T\to\infty} \eta_0 \circ \tau_{(T)}^1 = \eta_1$$

i.e., in the adiabatic limit our process induces the state transformation $\rho_i \to \rho_f$ on the system $\mathcal{S}$.

Improving Landauer's bound: Heat Full Statistics

Energetic Balance in average

Expected work done on the joint system $\mathcal{S} + \mathcal{R}$ (from Duhamel's formula):

$$\begin{aligned}
\langle \Delta W \rangle &= \lim_{T \to \infty} \int_0^1 \eta_0 \circ \tau_{(T)}^s (\dot{H}_{\mathcal{S}}(s) + \dot{\lambda}(s) V) \mathrm{d}s \\
&= \int_0^1 \eta_s (\dot{H}_{\mathcal{S}}(s) + \dot{\lambda}(s) V) \mathrm{d}s \\
&= \lim_{L \to \infty} \int_0^1 \frac{\mathrm{tr}\, \mathrm{e}^{-\beta H^{(L)}(s)} (\dot{H}^{(L)}(s))}{\mathrm{tr}\, \mathrm{e}^{-\beta H^{(L)}(s)}} \mathrm{d}s = F_f - F_i = \Delta F
\end{aligned}$$

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Expected change in energy of $\mathcal{S}$:

$$\langle \Delta U\rangle = \rho_f(H_{\mathcal{S}}(1)) - \rho_i(H_{\mathcal{S}}(0)) = -\beta^{-1}\Delta S + \Delta F$$

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Expected heat released in $\mathcal{R}$ (from the First Law!):

$$\langle \Delta Q\rangle = \langle \Delta W\rangle - \langle \Delta U\rangle = \beta^{-1}\Delta S$$

saturates Landauer's bound

Improving Landauer's bound: Heat Full Statistics

Full Statistics of dissipated heat

[Shimizu-Sakaki'91, Levitov-Lesovik'92, Kurchan'00, Tasaki'00, Avron-Bachmann-Graf-Grech'07. . .]

$$\mathbb{P}_{\mathrm{heat}}^{(L,T)}(\Delta Q) = \sum_{E_f - E_i = \Delta Q} \mathrm{tr}\left(P_{\{E_f\}}(H_{\mathcal{R}}^{(L)}) U_1^{(L,T)} P_{\{E_i\}}(H_{\mathcal{R}}^{(L)}) \eta_0^{(L)} P_{\{E_f\}}(H_{\mathcal{R}}^{(L)}) U_1^{(L,T)*}\right)$$

Improving Landauer's bound: Heat Full Statistics

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Cumulant generating function = Rényi relative entropy (use $H^{(L)}(0) = H_{\mathcal{R}}^{(L)} + \mathrm{const.}$)

$$\begin{aligned}
\chi_{\mathrm{heat}}^{(L,T)}(\alpha) &= \log \sum_{\Delta Q \in \mathrm{sp}(H_{\mathcal{R}}^{(L)}) - \mathrm{sp}(H_{\mathcal{R}}^{(L)})} \mathrm{e}^{-\alpha \Delta Q}\, \mathbb{P}_{\mathrm{heat}}^{(L,T)}(\Delta Q) \\
&= \log \mathrm{tr}\big(\mathrm{e}^{-\alpha H_{\mathcal{R}}^{(L)}} U_1^{(L,T)} \mathrm{e}^{\alpha H_{\mathcal{R}}^{(L)}} \eta_0^{(L)} U_1^{(L,T)*}\big) \\
&= \log \mathrm{tr}\big(\mathrm{e}^{-\alpha H^{(L)}(0)} U_1^{(L,T)} \mathrm{e}^{(\alpha-\beta) H^{(L)}(0)} U_1^{(L,T)*}\big) \\
&= \log \mathrm{tr}\big(\eta_0^{(L)\alpha/\beta} U_1^{(L,T)} \eta_0^{(L)(1-\alpha/\beta)} U_1^{(L,T)*}\big) \\
&= S_{\frac{\alpha}{\beta}}\big(\eta_0^{(L)} | U_1^{(L,T)} \eta_0^{(L)} U_1^{(L,T)*}\big)
\end{aligned}$$

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Assumption III (Thermodynamiic Limit)

$$\chi_{\text{heat}}^{(T)}(\alpha) = \lim_{L\to\infty} \chi_{\text{heat}}^{(L,T)}(\alpha) = S_{\frac{\alpha}{\beta}}(\eta_0 | \eta_0 \circ \tau_{(T)}^1)$$

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Large time limit follows from isothermal adiabatic theorem and previous remark:

$$\begin{aligned}\chi_{\rm heat}(\alpha) &= \lim_{T\to\infty} \chi_{\rm heat}^{(T)}(\alpha) = S_{\frac{\alpha}{\beta}}(\eta_0|\eta_1) \\ &= S_{\frac{\alpha}{\beta}}(\rho_i \otimes \nu_{\mathcal{R}}|\rho_f \otimes \nu_{\mathcal{R}}) = S_{\frac{\alpha}{\beta}}(\rho_i|\rho_f) = -\frac{\alpha}{\beta}\log d + \operatorname{tr} \rho_f^{1-\frac{\alpha}{\beta}}\end{aligned}$$

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Convergence of Heat Full Statistics

$$\mathbb{P}_{\rm heat}^{(L,T)} \Rightarrow \mathbb{P}_{\rm heat}^{(T)}\ (L\to\infty), \qquad \mathbb{P}_{\rm heat}^{(T)} \Rightarrow \mathbb{P}_{\rm heat}\ (T\to\infty)$$

with

$$\chi_{\rm heat}(\alpha) = \log \int {\rm e}^{-\alpha\Delta Q} {\rm d}\mathbb{P}_{\rm heat}(\Delta Q)$$

Improving Landauer's bound: Heat Full Statistics

While ΔQ is a discrete random variable for finite L, in the thermodynamic limit it acquires a continuous range. It becomes atomic again in the adiabatic limit.

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Denote by p_k and m_k the distinct eigenvalues of ρ_f and their multiplicities and set

$$Q_k = \frac{\log d + \log p_k}{\beta}$$

then

$$\mathbb{P}_{\mathrm{heat}}(\Delta Q = Q_k) = p_k m_k = \frac{m_k}{d} \mathrm{e}^{\beta Q_k}$$

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Under the same assumptions, a similar construction yields the Full Statistics of the work done on the joint system $\mathcal{S} + \mathcal{R}$

$$\chi_{\mathrm{work}}(\alpha) = -\alpha \Delta F$$

which shows that work does not fluctuate in the adiabatic limit

$$\mathrm{d}\mathbb{P}_{\mathrm{work}}(\Delta W) = \delta(\Delta W - \Delta F)\mathrm{d}\Delta W$$

it converges a.s. towards ΔF, a result which leads us to interpret ΔF as the change in free energy of the joint system $\mathcal{S} + \mathcal{R}$.

Improving Landauer's bound: The Perfect Erasure Limit

Until now $\rho_f > 0$. Perfect erasure aims at pure final state and can not be reached by coupling to a thermal reservoir at positive temperature. Consider the simplest case $d = 2$ with

$$\rho_f = (1 - \epsilon)|+\rangle\langle+| + \epsilon|-\rangle\langle-|, \qquad \epsilon \in]0, 1[$$

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Differentiating $\chi_{\text{heat}}(\alpha)$ at $\alpha = 0$ yields

$$\langle\Delta Q\rangle = \beta^{-1}\log 2 + \mathcal{O}(\epsilon\log\epsilon)$$

and higher cumulants

$$\langle\langle\Delta Q^n\rangle\rangle = \mathcal{O}(\epsilon(\log\epsilon)^n)$$

In the limit $\epsilon \to 0$

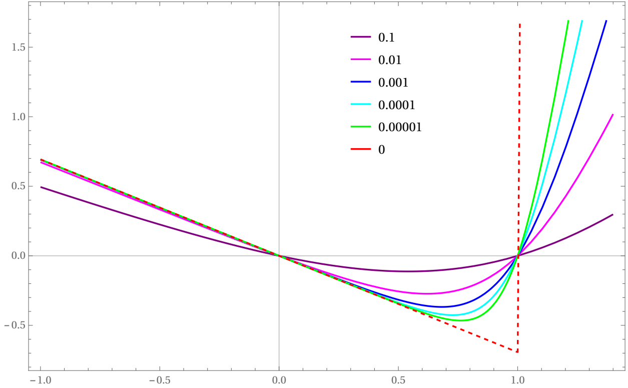
$$\mathbb{P}_{\text{heat}} \Rightarrow \delta_{\beta^{-1}\log 2}$$

but this weak convergence does not capture the singular nature of this limit: for small $\epsilon > 0$ there is a small but non-vanishing $\mathcal{O}(\epsilon)$ probability of violating Landauer's bound which is associated to failure of the erasure process

Improving Landauer's bound: The Perfect Erasure Limit

The limiting cumulant generating function is singular

$$\lim_{\epsilon \to 0} \chi_{\text{heat}}(\alpha) = \begin{cases} -\frac{\alpha}{\beta} \log d & \alpha < \beta \\ 0 & \alpha = \beta \\ +\infty & \alpha > \beta \end{cases}$$



Thank you !