Nonequilibrium Steady States of Open Quantum Systems

In Memory of Markus Büttiker

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Open System

A “small” system $\mathcal{S}$ interacts with a “large” reservoir $\mathcal{R}$

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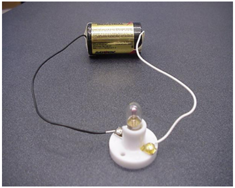
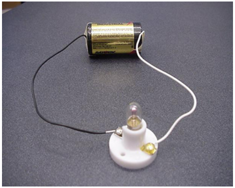
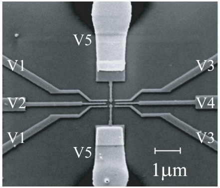


From macro ...

A “small” system $\mathcal{S}$ interacts with a “large” reservoir $\mathcal{R}$





From macro ... ... to nano ...

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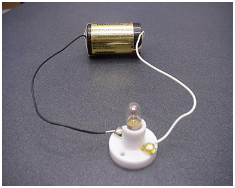
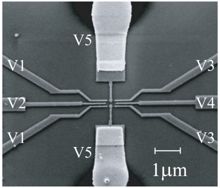
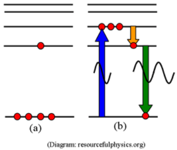


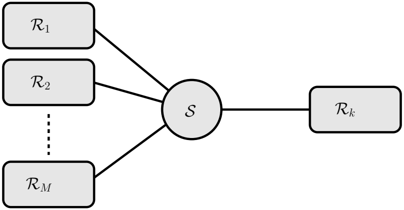




From macro ... ... to nano ... ... to micro

Steady State

$$\mathcal{R} = \mathcal{R}_1 + \cdots + \mathcal{R}_M$$



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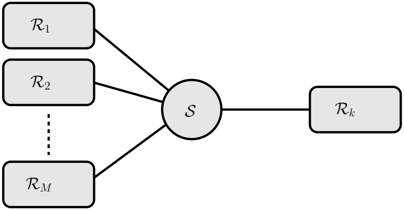


Physical point of view:

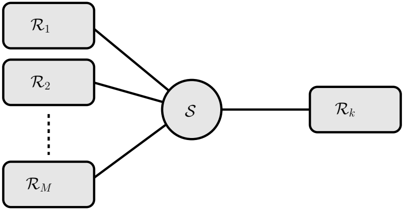
$\mathcal{R}$ can drive $\mathcal{S}$ to a steady regime (out of equilibrium) for some substantial time period

$$\downarrow$$

some globally conserved quantities are transported through the system $\mathcal{S}$
at constant rates

Steady State

$$\mathcal{R} = \mathcal{R}_1 + \cdots + \mathcal{R}_M$$



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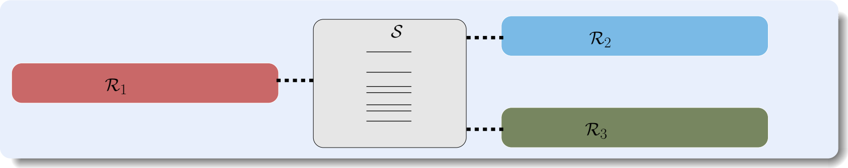
Mathematical idealization:

make $\mathcal{R}$ is infinitely extended and wait forever $\longrightarrow$ NESS

Models

Spin-Boson/Spin-Fermion/Pauli-Fierz

N-level atom interacting with free bosonic/fermionic fields at **positive density**

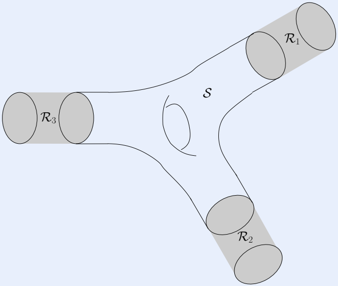


- Equilibrium [Hepp], [Hepp-Lieb], [Robinson], [Maassen], [Smedt-Dürr-Lebowitz-Liverani], [Jakšić-P], [Fidaleo-Liverani], [Bach-Fröhlich-Sigal], [Dereziński-Jakšić], [Fröhlich-Merkli-Sigal], [Fröhlich-Merkli], [Könenberg], ...
- NESS [Lebowitz-Spohn], [Jakšić-P,], [Merkli-Mück-Sigal], [de Roeck-Maes], [Dereziński-de Roeck-Maes], [Jakšić-Ogata-P] [de Roeck], [Jakšić-P-Westrich], [de Roeck-Kupiainen], ...

Models

Locally interacting quantum gas

Quantum gas at positive density on a geometric structure with **scattering ends**



Particles are non-interacting in the ends but may be interacting in the compact region $\mathcal{S}$ (electronics, photonics, spintronics, ...)

- Formal calculation of steady currents
  - Non-interacting case: **Büttiker-Landauer** scattering formalism
  - Interacting case: Keldysh formalism, Langreth rules, self-energy, ...
- Relaxation to NESS
  - Non-interacting case: Hilbert-space scattering theory
  - Interacting case: Hepp-Robinson-Ruelle algebraic scattering

Models

In this talk: simplest possible setup

- NESS of ideal and locally interacting quantum gases

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- Most results have been derived for (or are easily adapted to) continuous geometry under appropriate conditions (local decay estimates for one-particle dynamics & asymptotic completeness)

Models

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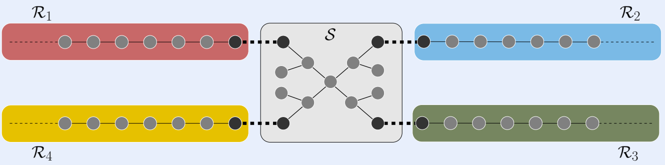
Electronic Black Box Model [EBB]

Fermi gas on a discrete structure (e.g., tight binding approximation)



Sample $\mathcal{S}$ connected to several leads $\mathcal{R}_1, \mathcal{R}_2, \ldots$

Electrons are non-interacting in the leads but may be interacting in the sample

The Büttiker-Landauer formalism

One-particle setup

- Configuration space: $\mathcal{S}$ finite set, $\mathcal{R}_j = \mathbb{Z}_+$
- Hilbert spaces: $\mathfrak{h}_\mathcal{S} = \ell^2(\mathcal{S})$, $\mathfrak{h}_j = \ell^2(\mathcal{R}_j)$

$$\mathfrak{h} = \mathfrak{h}_\mathcal{S} \oplus \mathfrak{h}_\mathcal{R}, \qquad \mathfrak{h}_\mathcal{R} = \oplus_j \mathfrak{h}_j$$

- Hamiltonians

$$h_\mathcal{S} = \sum_{x,y\in\mathcal{S}} h_\mathcal{S}(x,y)|x\rangle\langle y| \qquad h_j = c_j \sum_{\substack{x,y\in\mathcal{R}_j \\ |x-y|=1}} |x\rangle\langle y| \qquad h_\mathcal{R} = \sum_j h_j$$

$$\underbrace{h_{\boldsymbol{\tau}(t)} = \sum_j \tau_j(t)\left(|0_j\rangle\langle\varphi_j| + |\varphi_j\rangle\langle 0_j|\right)}_{\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots)} \qquad \underbrace{h_{\boldsymbol{\nu}(t)} = \sum_j \nu_j(t) 1_j}_{\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots)}$$

- Dynamics

$$\mathrm{i}\partial_t u(t,s) = (h_\mathcal{S} + h_\mathcal{R} + h_{\boldsymbol{\tau}(t)} + h_{\boldsymbol{\nu}(t)})u(t,s), \qquad u(s,s) = 1$$

Non-interacting EBB – Setup

Many-body setup

- $\mathcal{O}$ is the C^*-algebra generated by creation/annihilation operators a_x^*/a_x on the fermionic Fock space over $\mathfrak{h}$
$$\mathcal{O} = \mathcal{O}_{\mathcal{S}} \vee \mathcal{O}_{\mathcal{R}}$$
- Gauge group $\vartheta^t(a(f)) = a(\mathrm{e}^{\mathrm{i}t} f)$
- Observables are elements of $\mathcal{O}_\vartheta = \{A \in \mathcal{O} \,|\, \vartheta^t(A) = A, \text{ for all } t \in \mathbb{R}\}$
- Heisenberg dynamics $\alpha^{s,t}(a(f)) = a(u(t,s)^* f)$
- Reference state (gauge-invariant quasi-free/thermal equilibrium in each $\mathcal{R}_j$)

$$\langle a^*(f_1) \cdots a^*(f_n) a(g_m) \cdots a(g_1) \rangle_{\boldsymbol{\beta},\boldsymbol{\mu}} = \delta_{n,m} \det\{\langle g_i | \rho_{\boldsymbol{\beta},\boldsymbol{\mu}} f_j \rangle\}$$

$$\rho_{\boldsymbol{\beta},\boldsymbol{\mu}} = \left(\bigoplus_j \frac{1}{1 + \mathrm{e}^{\beta_j (h_j - \mu_j)}} \right) \oplus \frac{1}{2} 1_{\mathcal{S}} \qquad \begin{cases} \boldsymbol{\beta} = (\beta_1, \beta_2, \ldots) \\ \boldsymbol{\mu} = (\mu_1, \mu_2, \ldots) \end{cases}$$

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Definition [almost $(\boldsymbol{\beta},\boldsymbol{\mu})$-KMS states]

$\mathfrak{S}_{\boldsymbol{\beta},\boldsymbol{\mu}}$ is the set of gauge-invariant states on $\mathcal{O}$ which are normal w.r.t. the reference state $\langle\,\cdot\,\rangle_{\boldsymbol{\beta},\boldsymbol{\mu}}$, i.e., density matrices in the GNS representation of $\mathcal{O}$ induced by $\langle\,\cdot\,\rangle_{\boldsymbol{\beta},\boldsymbol{\mu}}$

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$$\rho_{\boldsymbol{\beta},\boldsymbol{\mu}} = \left(\bigoplus_j \frac{1}{1+\mathrm{e}^{\beta_j(h_j-\mu_j)}}\right) \oplus \frac{1}{2} 1_{\mathcal{S}} \qquad \left\{ \begin{array}{l} \boldsymbol{\beta} = (\beta_1, \beta_2, \ldots) \\ \boldsymbol{\mu} = (\mu_1, \mu_2, \ldots) \end{array} \right.$$

Remark

A state in $\mathfrak{S}_{\boldsymbol{\beta},\boldsymbol{\mu}}$ describes the situation where each reservoir $\mathcal{R}_j$ is near thermal equilibrium at inverse temperature β_j and chemical potential μ_j. This state is not necessarily quasi-free. In particular its restriction to the sample $\mathcal{S}$ can be any gauge-invariant state on $\mathcal{O}_{\mathcal{S}}$.

Keldysh correlators

Given an initial state $\langle \cdot \rangle_{\rm in}$ at time t_0, the Keldysh correlation functions of the system are defined by

$$G^{<}_{{\rm in},t_0}(t,s;f,g) = +{\rm i}\langle \alpha^{t_0,s}(a^*(g))\alpha^{t_0,t}(a(f))\rangle_{\rm in}$$
$$G^{>}_{{\rm in},t_0}(t,s;f,g) = -{\rm i}\langle \alpha^{t_0,t}(a(f))\alpha^{t_0,s}(a^*(g))\rangle_{\rm in}$$

A number of physically interesting quantities can be expressed in terms of these functions, e.g.,

- the local carrier density

$$n(x,t) = {\rm Im}\, G^{<}_{{\rm in},t_0}(t,t;\delta_x,\delta_x)$$

- the current at the junction $\mathcal{S} - \mathcal{R}_j$

$$\phi_j(t) = 2\tau_j {\rm Re}\, G^{<}_{{\rm in},t_0}(t,t;\delta_{0_j},\varphi_j)$$

- the local current at $x \in \mathcal{R}_j \cap [1,\infty[$

$$\phi_j(x,t) = 2c_j {\rm Re}\, G^{<}_{{\rm in},t_0}(t,t;\delta_{x-1},\delta_x)$$

Partitioned vs. Partition Free Scenario

The Partitioned Scenario (the traditional way)

- No bias: $\boldsymbol{\nu} = \mathbf{0}$
- Each subsystem is prepared independently
  - The sample is in some gauge-invariant state $\langle \cdot \rangle_{\mathcal{S}}$
  - Each reservoir $\mathcal{R}_j$ is in thermal equilibrium at inverse temperature β_j and chemical potential μ_j
- The resulting initial state $\langle \cdot \rangle_{\rm in}$ belongs to $\mathfrak{S}_{\boldsymbol{\beta},\boldsymbol{\mu}}$
- The subsystems are brought into contact at some early initial time $t_0 < 0$
  - either suddenly: $\boldsymbol{\tau}(t) = \boldsymbol{\tau} = (\tau_1, \tau_2, \ldots) = const.$
  - or adiabatically: $\boldsymbol{\tau}(t) = \chi(t/t_0)\boldsymbol{\tau}$
    ($\chi : \mathbb{R} \to [0, 1]$ continuous, $\chi(t) = 1$ for $t \leq 0$ and $\chi(t) = 0$ for $t \geq 1$)
- The state of the joint system at later time $t > t_0$ is

$$\langle A \rangle^{t}_{\boldsymbol{\beta},\boldsymbol{\mu},\mathbf{0}} = \langle \alpha^{t_0,t}(A) \rangle_{\rm in}$$

($\mathbf{0}$ refers to the absence of bias $\boldsymbol{\nu} = \mathbf{0}$)

Partitioned vs. Partition Free Scenario

The Partition Free Scenario [Cini]

- Constant coupling: $\boldsymbol{\tau}(t) = \boldsymbol{\tau} = (\tau_1, \tau_2, \ldots) = \mathit{const}$.
- The unbiased ($\boldsymbol{\nu} = \mathbf{0}$) coupled system $\mathcal{S} + \mathcal{R}$ is initially in thermal equilibrium at inverse temperature β and chemical potential μ
- The initial state $\langle \cdot \rangle_{\rm in}$ belongs to $\mathfrak{S}_{\boldsymbol{\beta},\boldsymbol{\mu}}$ with

$$\boldsymbol{\beta} = (\beta, \beta, \ldots) \qquad \boldsymbol{\mu} = (\mu, \mu, \ldots)$$

- At some early initial time $t_0 < 0$ the bias is switched on
  - either suddenly: $\boldsymbol{\nu}(t) = \boldsymbol{\nu} = (\nu_1, \nu_2, \ldots) = \mathit{const}$.
  - or adiabatically: $\boldsymbol{\nu}(t) = \chi(t/t_0)\boldsymbol{\nu}$
- The state of the joint system at later time $t > t_0$ is

$$\langle A \rangle^t_{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{\nu}} = \langle \alpha^{t_0,t}(A) \rangle_{\rm in}$$

Non-interacting EBB – NESS

Fix constants $\boldsymbol{\beta} \in \mathbb{R}_+^M$, $\boldsymbol{\mu}, \boldsymbol{\tau}, \boldsymbol{\nu} \in \mathbb{R}^M$ (assuming $\beta_j \neq \beta_k$ or $\beta_j(\mu_j + \nu_j) \neq \beta_k(\mu_k + \nu_k)$ for a pair $j \neq k$), and set

$$k_{\boldsymbol{\tau},\boldsymbol{\nu}} = h_{\mathcal{S}} + h_{\mathcal{R}} + h_{\boldsymbol{\tau}} + h_{\boldsymbol{\nu}}$$

Let

$$\omega_{\boldsymbol{\nu}} = \mathrm{s} - \lim_{t\to\infty} \mathrm{e}^{-\mathrm{i}tk_{\mathbf{0},\boldsymbol{\nu}}} \mathrm{e}^{\mathrm{i}tk_{\boldsymbol{\tau},\boldsymbol{\nu}}} p_{\mathrm{ac}}(k_{\boldsymbol{\tau},\boldsymbol{\nu}})$$

Denote $\mathcal{O}_+$ the C^*-algebra generated by $\{a(f) \,|\, f \in \mathfrak{h}_{\mathrm{ac}}(k_{\boldsymbol{\tau},\boldsymbol{\nu}})\}$. Then the Møller morphism defined by

$$\gamma_{\omega_{\boldsymbol{\nu}}}(a(f)) = a(\omega_{\boldsymbol{\nu}} f)$$

is a $*$-isomorphism $\mathcal{O}_+ \to \mathcal{O}_{\mathcal{R}}$.

Let the dynamics $\alpha^{t_0,t}$ be defined with

- either $\boldsymbol{\nu}(t) = \boldsymbol{\nu}$ and $\boldsymbol{\tau}(t) = \boldsymbol{\tau}$ (sudden coupling and biasing)
- or $\boldsymbol{\nu}(t) = \boldsymbol{\nu}$ and $\boldsymbol{\tau}(t) = \chi(t/t_0)\boldsymbol{\tau}$ (adiabatic coupling)
- or $\boldsymbol{\nu}(t) = \chi(t/t_0)\boldsymbol{\nu}$ and $\boldsymbol{\tau}(t) = \boldsymbol{\tau}$ (adiabatic biasing)

Non-interacting EBB – NESS

Then the following holds

Theorem

[Aschbacher-Jakšić-Pautrat-P], [Nenciu], [Cornean-Moldoveanu-P]

- For any $\langle\cdot\rangle_{\rm in}\in\mathfrak{S}_{\boldsymbol{\beta},\boldsymbol{\mu}}$, the limit

$$\langle A\rangle^{+}_{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{\nu}}=\lim_{T\to\infty}\frac{1}{T}\int_0^T\langle\alpha^{-t_0,0}(A)\rangle_{\rm in}\,{\rm d}t_0$$

  exists for all $A\in\mathcal{O}$ and defines a gauge-invariant state which is also invariant under the dynamics generated by $k_{\boldsymbol{\tau},\boldsymbol{\nu}}$
- If $A\in\mathcal{O}_+$ then

$$\langle A\rangle^{+}_{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{\nu}}=\lim_{t_0\to-\infty}\langle\alpha^{t_0,0}(A)\rangle_{\rm in}=\langle\gamma\omega_{\boldsymbol{\nu}}(A)\rangle_{\boldsymbol{\beta},\boldsymbol{\mu}}$$

  is the gauge-invariant quasi-free state with density $\omega^*_{\boldsymbol{\nu}}\rho_{\boldsymbol{\beta},\boldsymbol{\mu}}\omega_{\boldsymbol{\nu}}$

Non-interacting EBB – NESS

Theorem
[Aschbacher-Jakšić-Pautrat-P], [Nenciu], [Cornean-Moldoveanu-P]

- The NESS Keldysh correlators are given by

$$\begin{aligned} G^{<+}(t,s;f,g) &= \underbrace{\begin{matrix} +\mathrm{i}\langle \omega_{\boldsymbol\nu} f|\rho_{\boldsymbol\beta,\boldsymbol\mu}\mathrm{e}^{\mathrm{i}(t-s)k_{\mathbf{0},\boldsymbol\nu}}\omega_{\boldsymbol\nu} g\rangle \\ -\mathrm{i}\langle \omega_{\boldsymbol\nu} f|(1-\rho_{\boldsymbol\beta,\boldsymbol\mu})\mathrm{e}^{\mathrm{i}(t-s)k_{\mathbf{0},\boldsymbol\nu}}\omega_{\boldsymbol\nu} g\rangle \end{matrix}}_{\text{ac-part}} + \underbrace{\begin{matrix} G^{<+}_{\mathrm{pp}}(t-s;f,g) \\ G^{>+}_{\mathrm{pp}}(t-s;f,g) \end{matrix}}_{\text{pp-part}} \\ G^{>+}(t,s;f,g) &= \end{aligned}$$

- The "ac-part"

$$\mathcal{O}_+ \ni A \mapsto \langle A\rangle^+_{\boldsymbol\beta,\boldsymbol\mu,\boldsymbol\nu}$$

  of the NESS is independent of the initial state within $\mathfrak{S}_{\boldsymbol\beta,\boldsymbol\mu}$ and of the switching process (**no memory**)
- The "pp-part" depends on both the initial state and the history of the switching process

Non-interacting EBB – NESS

Let p_ε denote the spectral projections of $k_{\boldsymbol{\tau},\boldsymbol{\nu}}$

Theorem

- Partitioned scenario
  - Sudden coupling: [Aschbacher-Jakšić-Pautrat-P], [Nenciu], [Cornean-Moldoveanu-P]

$$G_{\rm pp}^{\gtrless+}(t,s;f,g)=\sum_{\varepsilon\in{\rm spec}_{\rm pp}(k_{\boldsymbol{\tau},\boldsymbol{\nu}})}{\rm e}^{-{\rm i}(t-s)\varepsilon}G_{{\rm in},0}^{\gtrless}(0,0;p_\varepsilon f,p_\varepsilon g)$$

  - Adiabatic coupling: [Cornean-Neidhardt-Zagrebnov] analyzed the dependence on the initial state, and expressed it in terms of adiabatic S-matrix
- Partition-free scenario
  - [Cornean-Duclos-Purice] proved that if the eigenvalues of $k_{\boldsymbol{\tau},\lambda\boldsymbol{\nu}}$ are real analytic for $\lambda\in[0,1]$ then

$$G_{\rm pp}^{\gtrless+}(t,s;f,g)=\sum_{\varepsilon\in{\rm spec}_{\rm pp}(k_{\boldsymbol{\tau},\boldsymbol{\nu}})}{\rm e}^{-{\rm i}(t-s)\varepsilon}G_{{\rm in},0}^{\gtrless}(0,0;up_\varepsilon f,up_\varepsilon g)$$

    where u is a unitary mapping eigenvectors of $k_{\boldsymbol{\tau},\boldsymbol{\nu}}$ to the corresponding eigenvectors of $k_{\boldsymbol{\tau},\mathbf{0}}$
  - [Cornean-Jensen-Nenciu] made a detailed analysis of a more specific model showing the intricate dependency on the history of the adiabatic switching if some eigenvalue of $k_{\boldsymbol{\tau},\boldsymbol{\nu}(t)}$ collides with its ac spectrum during the switching process

Non-interacting EBB – Stationary Currents

The observable $\Phi_j(L)$ describing the local current of the fully coupled+biased system at $L \in \mathcal{R}_j$ is the second-quantization of the total derivative

$$-\mathrm{i}[k_{\boldsymbol{\tau},\boldsymbol{\nu}}, \chi_{\mathcal{R}_j \cap [L,\infty[}(x)]$$

its NESS expectation

$$\phi_j^+ = \langle \Phi_j(L) \rangle^+_{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{\nu}} = 2c_j \mathrm{Re}\, G^{<+}(0,0;\delta_L,\delta_{L+1})$$

does not see the pp-part of the NESS and does not depend on L. It is given by the celebrated Büttiker-Landauer formula

Theorem
[Aschbacher-Jakšić-Pautrat-P], [Nenciu], [Cornean-Moldoveanu-P]

$$\phi_j^+ = \sum_k \int \left(\frac{1}{1+\mathrm{e}^{\beta_j(\varepsilon-\mu_j-\nu_j)}} - \frac{1}{1+\mathrm{e}^{\beta_k(\varepsilon-\mu_k-\nu_k)}} \right) \mathcal{T}^{\boldsymbol{\nu}}_{j,k}(\varepsilon)\, \frac{\mathrm{d}\varepsilon}{2\pi}$$

where $\mathcal{T}^{\boldsymbol{\nu}}_{j,k}(\varepsilon)$ is the transmission probability

$$\mathcal{T}^{\boldsymbol{\nu}}_{j,k}(\varepsilon) = \left| s_{j,k}(\varepsilon) - \delta_{j,k} \right|^2$$

$s(\varepsilon) = [s_{j,k}(\varepsilon)]$ being the on-shell S-matrix of the pair $(k_{\mathbf{0},\boldsymbol{\nu}}, k_{\boldsymbol{\tau},\boldsymbol{\nu}})$

Non-interacting EBB – Transient Currents

If the fully coupled+biased Hamiltonian $k_{\tau,\nu}$ has eigenvalues, then for $t > 0$ the transient current at $L \in \mathcal{R}_j$

$$\phi_j(L; t_0, t) = 2c_j \mathrm{Re}\, G^{<}_{\mathrm{in},t_0}(t, t; \delta_{L-1}, \delta_L)$$

has a quasi-periodic component which must be time-averaged to reach a steady value. However, this component is localized near the sample. This is the main content of the following

Theorem [Cornean-Gianesello-Zagrebnov]

- For all $t \geq t_0$,

$$\lim_{L\to\infty} \phi_j(L; t_0, t) = 0$$

- For all $L \geq 0$ and $t \geq 0$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \phi_j(L; -t_0, t)\, \mathrm{d}t_0 = \phi_j^+$$

- For all $t \geq 0$

$$\lim_{L\to\infty} \limsup_{t_0\to-\infty} |\phi_j(L; t_0, t) - \phi_j^+| = 0$$

Remarks and further works in non-iteracting EBB

- Needless to say, energy and heat currents can be treated in a completely similar way. Strict positivity of entropy production is easy to check using the BL formula
- Linearized versions of the Büttiker-Landauer formula (i.e., formula for the conductance) have also been derived by [Cornean-Jensen-Moldoveanu], [Cornean-Duclos-Nenciu-Purice]
- A Büttiker-Landauer formula for unitary scattering (with application to non-semi-bounded one-particle Hamiltonians) was obtained by [Cornean-Neidhardt-Wilhelm-Zagrebnov]
- Büttiker-Landauer formula for scattering by continuous geometric structures have been obtained by [Cornean-Duclos-Purice] using stationary scattering theory. Another approach, via time-dependent scattering and Mourre theory was followed by [Ben Sâad-P]
- A quantum version of the Eckmann-Young model of heat transport has been studied by [Jacquet] using the Büttiker-Landauer framework
- Measurement of potential differences within the system can done using some reservoirs with tunable thermodynamic parameters as probes. The solvability of the resulting consistency equations has been studied in [Jacquet-P]
- A special class of EBB models is obtained by coupling a finite Jacobi matrix of size L to two leads. The LB formula allows to express the steady current (and hence its L-dependence) in terms of transfer matrix. [Bruneau-Jakšić-P] relate the vanishing of the spectral density of the steady current to the divergence of the norm of the transfer matrix as $L \to \infty$.

Local Interactions

Locally interacting models are obtained by adding an interaction term ξW to the second quantization ot the one-particle Hamiltonian $k_{\boldsymbol{\tau}(t),\boldsymbol{\nu}(t)}$. The general structure of this interaction Hamiltonian is

$$W = \sum_n \frac{1}{n!} \sum_{x_1,\ldots,x_n \in \mathcal{S}} \Phi^{(n)}(x_1,\ldots,x_n) N_{x_1} \cdots N_{x_n}$$

where $N_x = a_x^* a_x$ and $\Phi^{(n)}$ is a real valued function which vanishes whenever two of its arguments coincide. A typical example is the 2-body interaction Hamiltonian

$$W = \frac{1}{2} \sum_{x,y \in \mathcal{S}} w(x,y) a_x^* a_x a_y^* a_y$$

The resulting dynamics on $\mathcal{O}$ is well defined. It can be expressed as a Dyson series

$$\alpha_\xi^{s,t}(A) = \sum_{n=0}^{\infty} (\mathrm{i}\xi)^n \int_{s \le s_1 \le \cdots \le s_n \le t} [\alpha_0^{s,s_1}(W), [\alpha_0^{s,s_2}(W), \cdots [\alpha_0^{s,s_n}(W), \alpha_0^{s,t}(A)] \cdots]] \, \mathrm{d}s_1 \cdots \mathrm{d}s_n$$

where α_0 is the non-interacting dynamics

Locally interacting EBB – Strategy

The basic strategy for proving the existence of a NESS in locally interacting systems is inspired from the non-interacting case. It is reminiscent of the early works of [Hepp] and [Robinson] in the 70'. It was used later by [Botvich-Malyshev] to investigate thermal equilibrium states, and more recently by [Ruelle] and [Fröhlich-Merkli-Ueltschi] in the context of NESS.

- Construct a multi-body Møller operator intertwining the interacting dynamics with the non-interacting one

$$\lim_{t_0\downarrow-\infty} \alpha_0^{0,t_0}\circ\alpha_\xi^{t_0,0}(A)=\sum_{n=0}^{\infty}(\mathrm{i}\xi)^n\int_{-\infty\le s_1\le\cdots\le s_n\le 0}[\alpha_0^{0,s_1}(W),[\alpha_0^{0,s_2}(W),\cdots[\alpha_0^{0,s_n}(W),A]\cdots]]\mathrm{d}s_1\cdots\mathrm{d}s_n$$

- The integrals on the r.h.s. extend to an infinite simplex of $\mathbb{R}^n$. They can not make sense if the non-interacting Hamiltonian $k_{\boldsymbol{\tau},\boldsymbol{\nu}}$ has eigenvalues.
- Assume $k_{\boldsymbol{\tau},\boldsymbol{\nu}}$ has no eigenvalues and the above limit exists in the norm of $\mathcal{O}$. Then it defines a $*$-morphism $\eta_\xi:\mathcal{O}\to\mathcal{O}$ such that $\eta_\xi\circ\alpha_\xi^{0,t}=\alpha_0^{0,t}\circ\eta_\xi$ and

$$\begin{aligned}\lim_{t_0\downarrow-\infty}\langle\alpha_\xi^{t_0,t}(A)\rangle_{\mathrm{in}}&=\lim_{t_0\downarrow-\infty}\langle\alpha_0^{t_0,t}(\alpha_0^{t,t_0}\circ\alpha_\xi^{t_0,t}(A))\rangle_{\mathrm{in}}\\&=\langle\eta_\xi(A)\rangle^+_{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{\nu}}\\&=\langle\gamma_{\omega_{\boldsymbol{\nu}}}\circ\eta_\xi(A)\rangle_{\boldsymbol{\beta},\boldsymbol{\mu}}=\langle A\rangle^+_{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{\nu},\xi}\end{aligned}$$

Locally interacting EBB – Strategy

To prove existence of the limit: show that there exists constants C and σ such that

$$\left\| \int_{-\infty \le s_1 \le \cdots \le s_n \le 0} [\alpha_0^{0,s_1}(W), [\alpha_0^{0,s_2}(W), \cdots [\alpha_0^{0,s_n}(W), a(f)] \cdots]] \mathrm{d}s_1 \cdots \mathrm{d}s_n \right\| \le C\sigma^n$$

This follows from

- A local decay estimate à la [Jensen-Kato], and [Komech-Kopylova-Kunze] for its lattice version, which yields

$$\int_0^\infty \max_{x,y \in \mathcal{S}} |\langle \delta_x | \mathrm{e}^{\mathrm{i}tk_{\tau,\nu}} \delta_y \rangle| \, \mathrm{d}t < \infty$$

  provided $k_{\tau,\nu}$ has no eigenvalue.
- The CAR algebra and a detailed combinatorial analysis due to [Botvich-Malyshev]. A particularly clear treatment of the combinatorics and an optimal formulation was more recently obtained by [Botvich-Maassen].

The following theorem of [Cornean-Moldoveanu-P] elaborates on the previous results of [Dirren-Fröhlich-Graf], [Fröhlich-Merkli-Ueltschi] and [Jakšić-Ogata-P].

Locally interacting EBB – NESS

Theorem

Assume that the fully coupled+biased one-particle Hamiltonian $k_{\boldsymbol{\tau},\boldsymbol{\nu}}$ has no eigenvalue (nor real resonance). Then there exists a constant $\xi_0 > 0$ such that, for $|\xi| < \xi_0$:

- The Møller morphism

$$\eta_\xi = \mathrm{s} - \lim_{t_0 \downarrow -\infty} \alpha_0^{0,t_0} \circ \alpha_\xi^{t_0,0}$$

  exists. Moreover, η_ξ is a $*$-automorphism of $\mathcal{O}$ and

$$\eta_\xi^{-1} = \mathrm{s} - \lim_{t_0 \downarrow -\infty} \alpha_\xi^{0,t_0} \circ \alpha_0^{t_0,0}$$

- For any initial state $\langle \cdot \rangle_{\mathrm{in}} \in \mathfrak{S}_{\boldsymbol{\beta},\boldsymbol{\mu}}$ the limit

$$\lim_{t_0 \downarrow -\infty} \langle \alpha_\xi^{t_0,0}(A) \rangle_{\mathrm{in}} = \langle \gamma_{\omega_{\boldsymbol{\nu}}} \circ \eta_\xi(A) \rangle_{\boldsymbol{\beta},\boldsymbol{\mu}} = \langle A \rangle^+_{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{\nu},\xi}$$

  exists and defines a gauge-invariant state on $\mathcal{O}$ which is invariant under the locally interacting dynamics $\alpha_\xi^{0,t}$. This state does not depend on the initial state, nor on the switching process.

Locally interacting EBB – Remarks & Complements

- The dynamical system $(\mathcal{O}, \alpha_\xi^{0,t}, \langle\cdot\rangle^+_{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{\nu},\xi})$ is mixing.
- [Fröhlich-Merkli-Ueltschi] gave a detailed perturbative analysis of the steady currents which, in particular, yields strict positivity of entropy production for small $|\xi|$. [Jakšić-P] showed that strict positivity is generic.
- In the unbiased case ($\boldsymbol{\nu} = \mathbf{0}$), the validity of linear response theory (Kubo formula and Onsager relations) has been established by [Jakšić-Ogata-P], and a central limit theorem for the NESS was proved by [Jakšić-Pautrat-P]. The linear response theory of biased system is still an open problem.
- For 2-body interactions, the Hartree-Fock approximation yields the NESS up to $\mathcal{O}(\xi^2)$. In particular, the corresponding BL formula gives the steady currents within this precision.
- The NESS Keldysh correlators $G^{\gtrless+}(t,s;\delta_x,\delta_y) = G^{\gtrless+}(t-s;\delta_x,\delta_y)$ are entire analytic functions of $t-s$, and

$$\lim_{t\to\pm\infty} \partial_z^n G^{\gtrless+}(z+t;\delta_x,\delta_y) = 0$$

  for all $z \in \mathbb{C}$ and $n \geq 0$. Their Fourier transform are continuous and there exists $\theta > 0$ such that

$$\sup_{\omega\in\mathbb{R}} \mathrm{e}^{\theta|\omega|}|\widehat{G}^{\gtrless+}(\omega;\delta_x,\delta_y)| < \infty$$

- Langreth rules reduce the calculation of steady state currents to that of the Keldysh correlator $G^{<+}(t;\varphi_j,\varphi_j)$.

Outlook

- Relaxation to NESS and the Büttiker-Landauer formalism are quite well understood for non-interacting continuous and discrete systems. However, bound states of the one-particle Hamiltonian induce hysteresis effects which need further and more systematic investigations.
- NESS of weakly interacting system are also accessible provided a good non-interacting approximation without bound state is available. This raises the question of extending the algebraic scattering approach to deal with point spectrum of the dynamical group (e.g. Abelian Møller morphisms?, adiabatic limit?).
- NESS are only one way of dealing with transport in a quantum gas. Other approaches have been investigated, e.g., the full counting statistics of charge transport [Avron-Bachman-Graf-Klich], [de Roeck], [Jakšić-Ogata-Pautrat-P], [Jakšić-Ogata-Seiringer-P] or quantum pumps (Büttiker-Prêtre formula) [Avron-Elgart-Graf-Sadun-Schnee]...
- Strongly correlated systems are not yet accessible to rigorous analysis (e.g. Kondo effect out of equilibrium), except for completely integrable models [Andrei-Mehta].
- Beyond Hartree-Fock: partial resummation of Dyson expansion.

Conclusion

There is still a huge gap between rigorous control of the dynamics and the formal Keldysh approach of daily use in physics.

Thank you !