What is Absolutely Continuous Spectrum ?

Based on joint works with
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Introduction

States of Motion of Particle Systems

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In a propagating states, particles escape any spatially bounded region as $|t| \to \infty$

Quantum Dynamics vs Spectral Analysis

Folklore from the 70' (atomic/molecular math-phys)

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$$i\partial_t \Psi_t = H\Psi_t, (\Psi_0 \in \mathcal{H}) \qquad \Longleftrightarrow \qquad \Psi_t = e^{-itH}\Psi_0$$

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[Sofer-Sigal '87, '94], [Dereziński '93], [. . . . . .] N-Body asymptotic completeness

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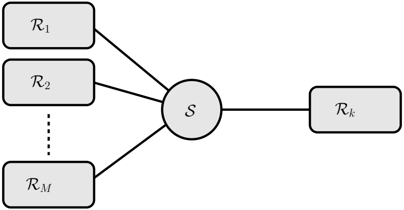
mathematical concept		physical quantity
ac spectrum	$\leftrightarrow$	**conductance**

Quantum Transport Theory vs Spectral Analysis ?

Transport in Non-Equilibrium Quantum Statistical Mechanics

A sample $\mathcal{S}$ (open system) driven by reservoirs ...



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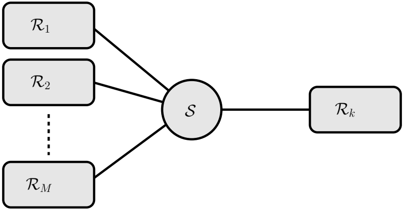

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Spectral Properties of the (bulk) sample Hamiltonian $h_{\mathcal{S}}$

$\updownarrow$

Large sample limit of the steady current

Program

spectral triple: $(\mathcal{H}, H, \Psi)$

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Jacobi matrix model $(\ell^2(\mathbb{N}), h, \delta_1)$

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$$h = h(a,b) = \begin{bmatrix} b_1 & a_1 & 0 & \\ a_1 & b_2 & a_2 & \ddots \\ 0 & a_2 & b_3 & \ddots \\ & \ddots & \ddots & \ddots \end{bmatrix}$$

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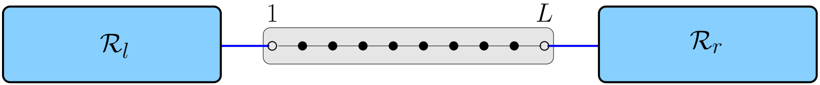
Transport properties of large truncated Jacobi matrix $h_S^{(L)} = 1_L h 1_L$ **on** $\ell^2([1..L])$



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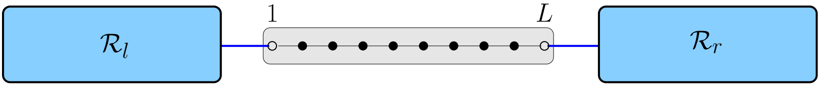

Physics

- Tranport properties and scaling theory of disordered 1D samples: Thouless, Anderson, Lee, Landauer,... ($\simeq$1970–1980)
- Scattering theory of steady state currents: Landauer, Büttiker, Fisher, Lee, Imry,... ($\simeq$1970–1990)

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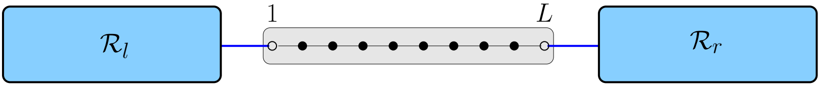


Mathematics

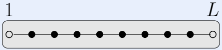
- Spectral theory of 1D Jacobi matrices . . . (1980–) [B. Simon's book 2011]
- Rigorous Landauer-Büttiker formalism: Aschbacher-Jakšić-Pautrat-P, Nenciu, Ben-Sâad-P (2005–2010)

Quasi-Free Quantum Transport

The model — One particle setup

The Sample

Bulk Hamiltonian: Jacobi matrix $h = h(a, b)$ on $\ell^2(\mathbb{N})$ ($b_n \neq 0$)
Sample Hamiltonian: $h_{\mathcal{S}}^{(L)}$ is the restriction of h to $\mathcal{H}_{\mathcal{S}}^{(L)} = \ell^2([1..L])$



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The Reservoirs

WLOG: $\mathcal{H}_{l/r} = L^2(\mathbb{R}, \mathrm{d}\nu_{l/r}(E))$, $h_{l/r} = E$, $\psi_{l/r} = 1$
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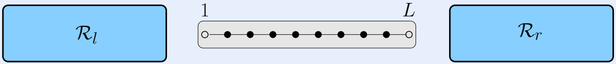

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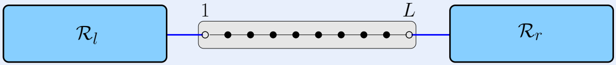
The Coupling

$$\mathcal{H}^{(L)} = \mathcal{H}_l \oplus \mathcal{H}_{\mathcal{S}}^{(L)} \oplus \mathcal{H}_r, \qquad h_0^{(L)} = h_l \oplus h_{\mathcal{S}}^{(L)} \oplus h_r, \qquad h^{(L)} = h_0^{(L)} + \kappa h_T$$

tunneling strength $\kappa \neq 0$, tunneling Hamiltonian

$$h_T = |\psi_l\rangle\langle 1| + |1\rangle\langle\psi_l| + |\psi_r\rangle\langle L| + |L\rangle\langle\psi_r|$$



The model — Many body setup

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- Steady state current

$$\begin{aligned} \langle J \rangle_{\mu_l,\mu_r,L} &= \lim_{T\to\infty} \frac{1}{T} \int_0^T \omega_{\mu_l,\mu_r}(\tau_t^{(L)}(J))\mathrm{d}t \\ &= \lim_{t\to\infty} \mathrm{TD} - \lim \left[\omega_{\mu_l,\mu_r}(N_l) - \omega_{\mu_l,\mu_r}(\tau_t^{(L)}(N_l))\right] \end{aligned}$$

Büttiker-Landauer vs Thouless conductance

The Büttiker-Landauer formula

The steady state current is given by

$$\langle J \rangle_{\mu_l,\mu_r,L} = \frac{1}{2\pi} \int_{\mu_l}^{\mu_r} \mathcal{T}^{(L)}(E) \mathrm{d}E$$

where

$$\mathcal{T}^{(L)}(E) = |S_{lr}(E)|^2 = 4\pi^2\kappa^4 |\langle 1|(h^{(L)} - E - \mathrm{i}0)^{-1}|L\rangle|^2 \frac{\mathrm{d}\nu_{l,\mathrm{ac}}}{\mathrm{d}E}(E) \frac{\mathrm{d}\nu_{r,\mathrm{ac}}}{\mathrm{d}E}(E)$$

is the sample's transmittance which satisfies the unitarity bound

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The proof involves the scattering theory of the pair $(h_0^{(L)}, h^{(L)})$

- Aschbacher, Jakšić, Pautrat, P.: JMP 48, 032101 (2007).
- Nenciu: JMP 48, 033302 (2007)
- Ben Sâad, P: JMP 55, 075202 (2014)

The Thouless conductance

Heuristics [Thouless 1977]

Consider an electron from the left reservoir on its journey towards the right reservoir. Let δt be the typical time such an electron spend in the sample. The time-energy uncertainty relation $\delta t \delta E \gtrsim 2\pi$ sets a limit on the spread in energy of its wave function: the Thouless energy

$$E_{\mathrm{Th}} = \delta E \gtrsim \frac{2\pi}{\delta t}.$$

Assuming a diffusive motion, we further have

$$L^2 = D\delta t$$

and Einstein's relation links the diffusion constant D to the conductivity σ

$$\sigma = D\varrho = \frac{L^2}{\delta t}\varrho \lesssim L^2 \frac{E_{\mathrm{Th}}}{2\pi}\varrho$$

where ϱ is the density of states of the sample. Denoting ΔE the typical level spacing of the sample, we have $\varrho L \Delta E \sim 1$. Thus, for the sample's conductance $g = \sigma / L$ we derive

$$g \lesssim g_{\mathrm{Th}} = \frac{1}{2\pi}\frac{E_{\mathrm{Th}}}{\Delta E}$$

g_{Th} is the Thouless conductance

The Thouless conductance

A tentative mathematical definition [Last 1994]

- For the sample's conductance to achieve its maximal value g_{Th}, the reservoir and its coupling should provide an optimal feeding of the sample with electrons.
- The coupling of the reservoirs to the sample should be *reflectionless*.
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$$h^{(3)}_{\text{crystal}} = \begin{bmatrix} \ddots & \ddots & & & & & & & \\ \kappa & b_1 & a_1 & & & & & & \\ & a_1 & b_2 & a_2 & & & & & \\ & & a_2 & b_3 & \kappa & & & & \\ & & & \kappa & b_1 & a_1 & & & \\ & & & & a_1 & b_2 & a_2 & & \\ & & & & & a_2 & b_3 & \kappa & \\ & & & & & & \ddots & \ddots & \ddots \end{bmatrix}$$

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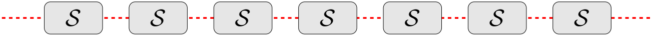

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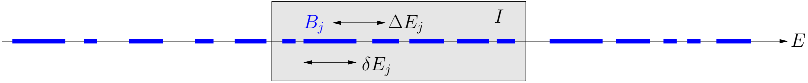

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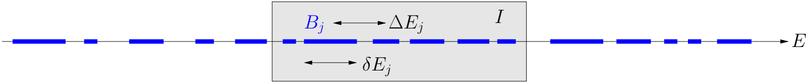

The energy uncertainty within a single band B_j is of the order of the bandwidth $\delta E_j = |B_j|$. A rough estimate of this uncertainty within I is

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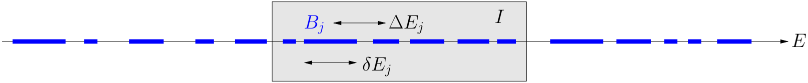

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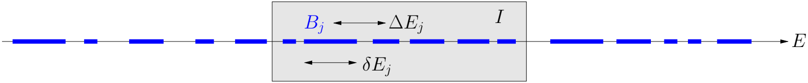

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The Thouless conductance is roughly the normalized Lebesgue measure of $\text{spec}(h^{(L)}_{\text{crystal}})$ in I

$$g_{Th} = \frac{1}{2\pi} \frac{\delta E}{\Delta E} \simeq \frac{1}{2\pi} \frac{|I \cap \text{spec}(h^{(L)}_{\text{crystal}})|}{|I|}$$

Crystaline reservoirs

Contrary to the Thouless conductance which is an intrinsic property of the sample, the Büttiker-Landauer conductance

$$g_{\mathrm{BL}}(\mu_l, \mu_r, L) = \frac{1}{2\pi(\mu_r - \mu_l)} \int_{\mu_l}^{\mu_r} \mathcal{T}^{(L)}(E)\mathrm{d}E$$

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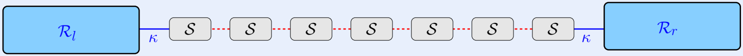

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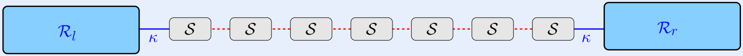

Theorem 1 [Bruneau, Jakšić, Last, P 2014]

$$\lim_{N\to\infty} g_{\mathrm{BL}}(\mu_l, \mu_r, L, N) = g_\infty(\mu_l, \mu_r, L)$$

$$\sup_{environment} g_\infty(\mu_l, \mu_r, L) = g_{Th}(]\mu_l, \mu_r[, L) = \frac{1}{2\pi} \frac{|\mathrm{spec}(h^{(L)}_{\mathrm{crystal}}) \cap]\mu_l, \mu_r[|}{|]\mu_l, \mu_r[|} \qquad (1)$$

where the supremum is taken over all realizations of the reservoirs/couplings.

Crystaline reservoirs

Contrary to the Thouless conductance which is an intrinsic property of the sample, the Büttiker-Landauer conductance

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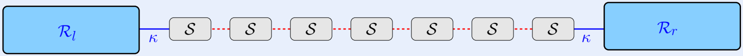

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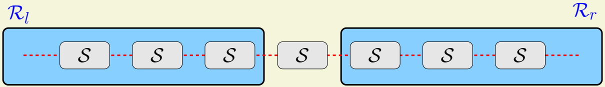

Physical vs mathematical characterization of conduction

Physical characterization

The Landauer-Büttiker formula naturally leads to the set $\mathcal{E}_{\text{conduction}}$ of energies E for which

$$\liminf_{L\to\infty} \lim_{\delta E\downarrow 0} g_{LB}(E-\delta E, E+\delta E, L) = \frac{1}{2\pi} \liminf_{L\to\infty} \mathcal{T}^{(L)}(E) > 0$$

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According to the folklore, let Σ_{bulk} denote the essential support of the absolutely continuous spectrum of h (i.e., of the original Hamiltonian H)

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Conjecture [Bruneau,Jakšić,P 2013]

$$\mathcal{E}_{\text{conduction}} = \Sigma_{\text{bulk}} \cap \Sigma_l \cap \Sigma_r$$

Remark. The $\cap\Sigma_l \cap \Sigma_r$ is trivially needed since $\mathcal{T}^{(L)}(E) = 0$ for $E \notin \Sigma_l \cap \Sigma_r$.

The Schrödinger conjecture

In terms of the Transfer matrix of h

$$T(E,L) = \begin{bmatrix} (E-b_L)/a_L & -1/a_L \\ a_L & 0 \end{bmatrix} \cdots \begin{bmatrix} (E-b_1)/a_1 & -1/a_1 \\ a_1 & 0 \end{bmatrix}$$

we have

Theorem 2 [Bruneau,Jakšić,P 2013]

$$\mathcal{E}_{\text{conduction}} = \{E \mid \sup_L \|T(E,L)\| < \infty\} \cap \Sigma_l \cap \Sigma_r \tag{2}$$

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[Gilbert-Pearson '87] proved that the rhs of (2) is included in Σ_{bulk}. Thus, our conjecture reduces to the reverse inclusion $\Sigma_{\text{bulk}} \cap \Sigma_l \cap \Sigma_r \subset \mathcal{E}_{\text{conduction}}$ which is equivalent to the celebrated (see the review [Maslov-Molchanov-Gordon, Russian J. Math. Phys. '93])

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$$\Sigma_{\text{bulk}} = \{E \mid \sup_L \|T(E,L)\| < \infty\}$$

which was believed to be true until Artur Avila succeeded in constructing an ergodic Schrödinger operator h which (with probability 1) has unbounded generalized eigenfunctions for a subset of positive Lebesgue measure of Σ_{bulk} [JAMS 28, 579–616 (2015)]

AC spectrum and conductance, finally

The main result of the our last paper is the following complete dynamical characterization of the ac-spectrum of h.

Theorem 3 [Bruneau,Jakšić,Last,P 2015]

Assume that $]\mu_l, \mu_r[\subset \Sigma_l \cap \Sigma_r$. Then the following statements are equivalent:

1. $\mathrm{spec}_{\mathrm{ac}}(h) \cap]\mu_l, \mu_r[= \emptyset$
2. $\lim_{L\to\infty} g_{LB}(\mu_l, \mu_r, L) = 0$
3. $\lim_{L\to\infty} g_{Th}(\mu_l, \mu_r, L) = 0$

Moreover, if $\mathrm{spec}_{\mathrm{ac}}(h) \cap]\mu_l, \mu_r[\neq \emptyset$, then

$$\liminf_{L\to\infty} g_{LB}(\mu_l, \mu_r, L) > 0, \qquad \liminf_{L\to\infty} g_{Th}(\mu_l, \mu_r, L) > 0$$

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Remarks.

- 1 $\Leftrightarrow$ 3 was proved in Last's PhD thesis in the ergodic case. Gestezsy-Simon extended Last's result to deterministic full line operators. Their argument does not work for the half line operators.
- Only 2 $\Rightarrow$ 1 requires the assumption $]\mu_l, \mu_r[\subset \Sigma_l \cap \Sigma_r$ which ensures that the interval $]\mu_l, \mu_r[$ is entirely open to scattering.

Ideas of the proof

Main strategy: show that ❶, ❷ and ❸ are equivalent to

⓿ $$\lim_{L\to\infty}\int_{\mu_l}^{\mu_r}\|T(E,L)\|^{-2}\mathrm{d}E = 0$$

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- The key lemma ① ⇒ ⓪ is a simple consequence of a result of Carmona, Krutikov-Remling and Simon: If $u = (1,0)^T$, then

$$\lim_{L\to\infty} \frac{1}{\pi} \int f(E) \|T(E,L)u\|^{-2} \mathrm{d}E = \langle 1 | f(h) | 1 \rangle$$

  for $f \in C_0(\mathbb{R})$.

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- The reverse implication ⓪ $\Rightarrow$ ❶ follows from the Simon-Last result:

$$\liminf_{L\to\infty} \|T(E,L)\| < \infty$$

  for a.e. $E \in \Sigma_{\mathrm{bulk}}$

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- The equivalence ⓪ ⇔ ❷ follows from Theorem 2.
- The proof of ❸ ⇒ ⓪ is essentially Last's purely deterministic proof of ❸ ⇒ ❶
- Last's proof of ❶ ⇒ ❸ relies on Kotani theory which yields the estimate

$$\limsup_{L\to\infty} |\mathrm{spec}_{\mathrm{ac}}(h^{(L)}_{\mathrm{crystal}}) \cap]\mu_l, \mu_r[| \leq |\mathrm{spec}_{\mathrm{ac}}(h) \cap]\mu_l, \mu_r[|$$

with probability 1. Combining Last's deterministic estimate of $\|T(E,L)\|$ for $E \in \mathrm{spec}(h^{(L)}_{\mathrm{crystal}})$ with a result of Deift-Simon on the rotation number of $h^{(L)}_{\mathrm{crystal}}$, we derive the estimate

$$\limsup_{L\to\infty} |\mathrm{spec}_{\mathrm{ac}}(h^{(L)}_{\mathrm{crystal}}) \cap]\mu_l, \mu_r[| \leq C |\mathrm{spec}_{\mathrm{ac}}(h) \cap]\mu_l, \mu_r[|^{1/5}$$

which yields ⓪ ⇒ ❸

Outlook

Many-body localization

Suppose $\{v(x)\}_{x\in\mathbb{Z}_+}$ are "nice" i.i.d. random variables. Then *h* has pure point spectrum and Theorem 3 implies

$$\limsup_{L\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \omega_{\mu_l,\mu_r}(\tau_t^{(L)}(J))\mathrm{d}t = 0$$

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Consider adding short-range many-body interactions to the second quantized Hamiltonian

$$H^{(L)} = \Gamma(h^{(L)}) + W, \qquad W = \sum_{x,y\in[0..L]} w(x-y)a_x^* a_y^* a_y a_x$$

The dynamics on $\mathrm{CAR}(\mathcal{H}^{(L)})$ is still well defined

$$\tau_t^{(L)}(A) = \mathrm{e}^{\mathrm{i}tH^{(L)}} A \mathrm{e}^{-\mathrm{i}tH^{(L)}}$$

so is the current observable

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Does Anderson localization survive weak many-body interactions ? How to characterize it ?

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We can still characterize localization/conduction on $]\mu_l, \mu_r[$ by

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Anderson impurity model

The simplest model in this category (repulsive interaction between electrons at $x = 1$ and $x = 2$)

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Even this simplest model is completely open!

Papers

- Pioneering work: ”Conductance and Spectral Properties”. Yoram Last, unpublished PhD thesis (1994)

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- Bruneau, Jakšić, P: Landauer-Büttiker formula and Schrödinger conjecture. CMP 319, 501–513 (2013)
- Jakšić, Landon, P: Entropic fluctuations in XY chains and reflectionless Jacobi matrices. AHP 14, 1775–1800 (2013)
- Jakšić, Landon, Panati: A note on reflectionless Jacobi matrices. CMP 332, 827–838 (2014)
- Bruneau, Jakšić, Last, P: Landauer-Büttiker and Thouless conductance. CMP 338, 347–366 (2015)
- Bruneau, Jakšić, Last, P: Conductance and absolutely continuous spectrum of 1D samples. CMP 344, 959–981 (2016).
- Bruneau, Jakšić, Last, P: Crystalline Conductance and Absolutely Continuous Spectrum of 1D Samples. LMP 106, 0787–797 (2017).
- Bruneau, Jakšić, Last, P: What is absolutely continuous spectrum ? Proc. ICMP 2015.

Thank you !