Tomita–Takesaki theory

JAN DEREZIŃSKI[1], CLAUDE-ALAIN PILLET[2]

[1] Department of Mathematical Methods in Physics
Warsaw University
Hoża 74, 00-682, Warsaw, Poland
jan.derezinski@fuw.edu.pl

[2] CPT-CNRS, UMR 6207
Université du Sud Toulon-Var
B.P. 20132
83957 La Garde Cedex, France
pillet@univ-tln.fr

1 Modular states

Let $\mathfrak{M}$ be a von Neumann algebra acting on the Hilbert space $\mathcal{H}$. Its commutant

$$\mathfrak{M}' = \{A \in \mathcal{B}(\mathcal{H}) \,|\, AB = BA \text{ for all } B \in \mathfrak{M}\}$$

is also a von Neumann algebra. Von Neumann's bicommutant theorem states that $\mathfrak{M}'' = \mathfrak{M}$.

Definition 1 *A vector $\Psi \in \mathcal{H}$ is cyclic for $\mathfrak{M}$ if the subspace $\mathfrak{M}\Psi$ is dense in $\mathcal{H}$. It is separating for $\mathfrak{M}$ if $A\Psi = 0$ for some $A \in \mathfrak{M}$ implies $A = 0$. It is modular if it is both cyclic and separating for $\mathfrak{M}$.*

A vector $\Psi \in \mathcal{H}$ is separating for $\mathfrak{M}$ if and only if the corresponding normal state $\omega_\Psi(A) = (\Psi|A\Psi)$ is faithful.

The support s_ω of a normal state ω on $\mathfrak{M}$ is the smallest orthogonal projection $P \in \mathfrak{M}$ such that $\omega(P) = 1$. It follows that $\omega(A^*A) = 0$ if and only if $As_\omega = 0$. In particular, ω is faithful if and only if $s_\omega = I$.

Lemma 2 *The support of the vector state ω_Ψ is the orthogonal projection on the closure of $\mathfrak{M}'\Psi$. Consequently, a vector $\Psi \in \mathcal{H}$ is separating for $\mathfrak{M}$ if and only if it is cyclic for $\mathfrak{M}'$.*

Remark. Since $\mathfrak{M}'' = \mathfrak{M}$, it follows that Ψ is cyclic for $\mathfrak{M}$ if and only if it is separating for $\mathfrak{M}'$.

Let $\mathcal{O}$ be a C^*-algebra and ω a state on $\mathcal{O}$. Denote by $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ the GNS representation of $\mathcal{O}$ induced by ω.

Definition 3 *The state ω is modular if the vector Ω_ω is modular for the enveloping von Neumann algebra $\mathcal{O}_\omega = \pi_\omega(\mathcal{O})''$.*

Note that the state ω is modular if and only if the vector state induced by Ω_ω is faithful on $\mathcal{O}_\omega$. This does not imply, nor is it implied by the faithfulness of ω. However, if $\mathcal{O}$ is a von Neumann algebra then a faithful normal state ω is modular.

The following result links modular theory with the theory of KMS states. It is often useful in applications to statistical mechanics.

Theorem 4 *Let $(\mathcal{O}, \tau)$ be a C^*- or W^*-dynamical system. Any (τ, β)-KMS state, $\beta \in \mathbb{R}$, is modular.*

2 Modular structure

Let Ψ be a modular vector for $\mathfrak{M}$. Since Ψ is separating for $\mathfrak{M}$

$$A\Psi \mapsto A^*\Psi,$$

defines an anti-linear involution S_0 of $\mathfrak{M}\Psi$. Inspection of the graph of S_0 and the fact that Ψ is cyclic for $\mathfrak{M}'$ show that S_0 is closable and that its closure S is involutive. Since Ψ is cyclic for $\mathfrak{M}$, S is densely defined and hence has a densely defined adjoint $S^* = S_0^*$. Define the self-adjoint operator $\Delta = S^*S$ and write the polar decomposition of S as $S = J\Delta^{1/2}$. Since S is injective and has dense range J is anti-unitary. From $I = S^2 = J\Delta^{1/2}J\Delta^{1/2}$ we conclude that $J\Delta^{1/2} = \Delta^{-1/2}J^*$. It follows that $J^2\Delta^{1/2} = J\Delta^{-1/2}J^*$ and the unicity of the polar decomposition yields $J^2 = I$, i.e., $J = J^*$.

Definition 5 *The positive self-adjoint operator Δ is the modular operator and the anti-unitary involution J the modular conjugation of the pair $(\mathfrak{M}, \Psi)$.*

The deep algebraic properties of the modular operator and conjugation are the content of Tomita-Takesaki's theorem:

Theorem 6 *Let Ψ be a modular vector for the von Neumann algebra $\mathfrak{M}$. If Δ and J are the corresponding modular operator and modular conjugation then the following hold:*

1. $J\mathfrak{M}J = \mathfrak{M}'$.
2. *For any $t \in \mathbb{R}$ one has $\Delta^{\mathrm{i}t}\mathfrak{M}\Delta^{-\mathrm{i}t} = \mathfrak{M}$.*

Due to the unbounded nature of S the proof of this theorem is technically involved. It was first published in [T] but more compact expositions can be found e.g. in [BR1]. A technically simpler proof is [RVD].

Definition 7 *The group of $*$-automorphisms of $\mathfrak{M}$ defined by $\sigma^t(A) = \Delta^{\mathrm{i}t}A\Delta^{-\mathrm{i}t}$ is the modular group of the pair $(\mathfrak{M}, \Psi)$.*

More generally, if ω is a faithful normal state on the von Neumann algebra $\mathcal{O}$ and Δ the modular operator of $(\pi_\omega(\mathcal{O}), \Omega_\omega)$ then $\sigma_\omega^t(A) = \pi_\omega^{-1}(\Delta^{\mathrm{i}t}\pi_\omega(A)\Delta^{-\mathrm{i}t})$ is the modular group of ω.

The main property of the modular group is the following result due to Takesaki which can be seen as a reverse of Theorem 4.

Theorem 8 *Let ω be a faithful normal state on the von Neumann algebra $\mathcal{O}$. Then ω is a KMS state for the modular group σ_ω at inverse temperature $\beta = -1$. Moreover, the modular group is the only dynamics on $\mathcal{O}$ for which ω has this property.*

The modular conjugation allows to construct another central object of modular theory.

Definition 9 *The natural cone associated to the pair $(\mathfrak{M}, \Psi)$ is the closed subset of $\mathcal{H}$ defined by*

$$\mathcal{H}_+ = \{AJAJ\Psi \,|\, A \in \mathfrak{M}\}^{\mathrm{cl}}.$$

The most important properties of the natural cone are the following.

Theorem 10 *The natural cone $\mathcal{H}_+$ is self-dual, i.e.,*

$$\mathcal{H}_+ = \widehat{\mathcal{H}}_+ \equiv \{\Omega \in \mathcal{H} \,|\, (\Phi \,|\, \Omega) \geq 0 \text{ for all } \Phi \in \mathcal{H}_+\}.$$

In particular, $\mathcal{H}_+$ is convex. Moreover, the following hold:

1. $J\Phi = \Phi$ *for all* $\Phi \in \mathcal{H}_+$.
2. $AJA\mathcal{H}_+ \subset \mathcal{H}_+$ *for all* $A \in \mathfrak{M}$.
3. $JAJ = A^*$ *for all* $A \in \mathfrak{M} \cap \mathfrak{M}'$.

3 Standard representation

Definition 11 *A quadruple $(\mathcal{H}, \pi, J, \mathcal{H}_+)$ is a standard representation of the W^*-algebra $\mathfrak{M}$ if $\pi : \mathfrak{M} \to \mathcal{B}(\mathcal{H})$ is a representation of $\mathfrak{M}$, J an antiunitary involution on $\mathcal{H}$ and $\mathcal{H}_+$ a self-dual cone in $\mathcal{H}$ satisfying the following conditions:*

1. *$J\pi(\mathfrak{M})J = \pi(\mathfrak{M})'$;*
2. *$J\pi(A)J = \pi(A)^*$ for all A in the center of $\mathfrak{M}$;*
3. *$J\Psi = \Psi$ for all $\Psi \in \mathcal{H}_+$;*
4. *$\pi(A)J\pi(A)\mathcal{H}_+ \subset \mathcal{H}_+$ for all $A \in \mathfrak{M}$.*

One of the key result in the theory of W^*-algebras is the following.

Theorem 12 *Any W^*-algebra $\mathfrak{M}$ has a faithful standard representation. Moreover, this representation is unique, up to unitary equivalence.*

If $\mathfrak{M}$ is separable it has a faithful normal state ω and Theorem 10 shows that the corresponding GNS representation is standard. This is in particular the case of von Neumann algebras over separable Hilbert spaces which are most often encountered in physical applications. See [SZ] for the general case.

The standard representation has two properties which are of crucial importance in the study of quantum dynamical systems. The first one deals with normal states.

Theorem 13 *Let $(\mathcal{H}, \pi, J, \mathcal{H}_+)$ be a standard representation of $\mathfrak{M}$. Any normal state ω on $\mathfrak{M}$ has a unique vector representative $\Phi_\omega \in \mathcal{H}_+$ such that $\omega(A) = (\Phi_\omega|\pi(A)\Phi_\omega)$. Moreover,*

$$\|\Phi_\omega - \Phi_\nu\| \le \|\omega - \nu\| \le \|\Phi_\omega - \Phi_\nu\| \, \|\Phi_\omega + \Phi_\nu\|,$$

holds for all normal states ω, ν. Thus, there is an homeomorphic correspondence between normal states and unit vectors of $\mathcal{H}_+$. Finally, $(\pi(\mathfrak{M})\Phi_\omega)^{\rm cl} = J\,(\pi(\mathfrak{M})'\Phi_\omega)^{\rm cl}$, and in particular

$$\omega \text{ is faithful} \Leftrightarrow \Phi_\omega \text{ is separating for } \pi(\mathfrak{M}) \Leftrightarrow \Phi_\omega \text{ is cyclic for } \pi(\mathfrak{M}).$$

The second property concerns the unitary implementation of $*$-automorphisms of $\mathfrak{M}$ in a standard representation $(\mathcal{H}, \pi, J, \mathcal{H}_+)$. Denote by $\mathrm{Aut}(\mathfrak{M})$ the topological group of $*$-automorphisms of $\mathfrak{M}$ with the topology of pointwise σ-weak convergence. Let also $\mathcal{U}$ be the set of unitaries of $\mathcal{H}$ such that $U\pi(\mathfrak{M})U^* = \pi(\mathfrak{M})$ and $U\mathcal{H}_+ \subset \mathcal{H}_+$. Equipped with the strong operator topology $\mathcal{U}$ is a topological group and $\tau_U(A) = \pi^{-1}(U\pi(A)U^*)$ defines a continuous morphism $\mathcal{U} \to \mathrm{Aut}(\mathfrak{M})$.

Theorem 14 *The map $U \mapsto \tau_U$ is a topological isomorphism. Moreover, for any $U \in \mathcal{U}$ and any normal state ω on $\mathfrak{M}$ one has*

1. *$JUJ = U$.*
2. *$U\pi(\mathfrak{M})'U^* = \pi(\mathfrak{M})'$.*
3. *$U^*\Phi_\omega = \Phi_{\omega\circ\tau_U}$.*

In particular, if $(\mathfrak{M}, \tau)$ is a W^*-dynamical system then there exists a unique self-adjoint operator L on $\mathcal{H}$ such that $\pi(\tau^t(A)) = \mathrm{e}^{\mathrm{i}tL}\pi(A)\mathrm{e}^{-\mathrm{i}tL}$ and $\mathrm{e}^{\mathrm{i}tL}\mathcal{H}_+ \subset \mathcal{H}_+$.

Definition 15 *The generator L is called standard Liouvillean of the dynamical system $(\mathfrak{M}, \tau)$.*

The standard Liouvillean is uniquely defined up to unitary equivalence. If ω is a modular τ-invariant state then the induced GNS representation is standard and the ω-Liouvillean (see Section 3 in [Quantum dynamical systems]) coincide with the standard Liouvillean. This is in particular the case if ω is a KMS state for τ.

The spectral properties of the standard Liouvillean are intimately related to the properties of the corresponding dynamical system. As an illustration, the following result is a direct consequence of Theorem 13 (see [Quantum Koopmanism] for more information on this subject).

Theorem 16 *Let L be the standard Liouvillean of the W^*-dynamical system $(\mathfrak{M}, \tau)$.*

1. *L has no eigenvalues if and only if there is no normal τ-invariant state on $\mathfrak{M}$.*
2. *$\mathrm{Ker}(L)$ is one-dimensional if and only if there is a unique normal τ-invariant state ω on $\mathfrak{M}$. In this case Φ_ω is the unique unit vector in $\mathrm{Ker}(L) \cap \mathcal{H}_+$.*

4 The finite dimensional case

It is instructive to work out the standard representation of a finite dimensional von Neumann algebra $\mathfrak{M} \subset \mathcal{B}(\mathbb{C}^N)$. This case is particularly simple since $\mathcal{B}(\mathbb{C}^N)$ is itself a Hilbert space for the inner product $(X|Y) = \mathrm{tr}(X^*Y)$. One has $\mathcal{B}(\mathbb{C}^N) = \mathfrak{M} \oplus \mathfrak{M}^\perp$ and the predual $\mathfrak{M}_*$ can be identified with $\mathfrak{M}$. Since any $A \in \mathfrak{M}$ can be written as a linear combination of 4 non-negative elements of $\mathfrak{M}$ it is easy to see that there exists a basis $\rho_1, \ldots, \rho_n$ of $\mathfrak{M}$ such that $\rho_j \geq 0$ and $\mathrm{tr}\,\rho_j = 1$. It follows that

$$\omega = \frac{1}{n} \sum_{j=1}^{n} \rho_j,$$

defines a faithful state on $\mathfrak{M}$. Consider $\mathcal{H} = \mathfrak{M}$ as a Hilbert space (a subspace of $\mathcal{B}(\mathbb{C}^N)$). Then $\Omega = \omega^{1/2}$ is a unit vector in $\mathcal{H}$. Moreover, the map $\pi : \mathfrak{M} \to \mathcal{B}(\mathcal{H})$ defined by $\pi(A)X = AX$ is a $*$-morphism such that $\omega(A) = (\Omega|\pi(A)\Omega)$. Denote by P the orthogonal projection on $(\mathrm{Ker}\,\omega)^\perp$. Clearly $P \in \mathfrak{M}$ and since $\rho_j \geq 0$ one has $\mathrm{Ker}\,\omega \subset \cap_j \mathrm{Ker}\,\rho_j$ and hence $\mathrm{Ker}\,\omega \subset \mathrm{Ker}\,A$ for all $A \in \mathfrak{M}$. It follows that $\mathrm{Ran}\,A \subset \mathrm{Ran}\,P$ and hence $A = PA$ for all $A \in \mathfrak{M}$. Since the last identity is equivalent to $A^* = A^*P$ we conclude that $A = AP = PA = PAP$ for all $A \in \mathfrak{M}$, i.e., that P is the unit of $\mathfrak{M}$. Since there exists $T \in \mathfrak{M}$ such that $T\omega^{1/2} = P$ we can write $\pi(XT)\Omega = X$ and conclude that $\pi(\mathfrak{M})\Omega = \mathcal{H}$. We have shown that $(\mathcal{H}, \pi, \Omega)$ is the GNS representation of $\mathfrak{M}$ induced by ω. Note that since ω is a normal faithful state, this representation is itself faithful.

The formulas $JX = X^*$ and $\Delta^{1/2}X = \omega^{1/2}XT$ define an anti-unitary involution and a positive self-adjoint operator on $\mathcal{H}$ such that

$$J\Delta^{1/2}\pi(A)\Omega = \pi(A)^*\Omega.$$

Thus, J and Δ are the modular conjugation and the modular operator of the pair $(\pi(\mathfrak{M}), \Omega)$.

Elements of the natural cone are given by $\pi(A)J\pi(A)\Omega = A\omega^{1/2}A^*$ from which we can conclude that

$$\mathcal{H}_+ = \{A \in \mathfrak{M} \,|\, A \geq 0\},$$

and one easily checks the validity of Theorem 10.

Let C be an element of $\pi(\mathfrak{M})'$. For all $A \in \mathfrak{M}$ and $X \in \mathcal{H}$ one has

$$C(AX) = C(\pi(A)X) = \pi(A)(C(X)) = AC(X).$$

Setting $X = P$, the unit of $\mathfrak{M}$, and $B = C(P) \in \mathfrak{M}$, we get $C(A) = AB$. We conclude that $\pi(\mathfrak{M})'$ consists of the linear maps $X \mapsto XB$ with $B \in \mathfrak{M}$. Thus, $J\pi(\mathfrak{M})J = \pi(\mathfrak{M})'$ and we have obtained the standard representation of the finite dimensional von Neumann algebra $\mathfrak{M}$.

Any normal state ν on $\mathfrak{M}$ is given by $\nu(A) = \mathrm{tr}(\rho A)$ for a density matrix $\rho \in \mathfrak{M}$. It follows that $\nu \mapsto \Phi_\nu = \rho^{1/2} \in \mathcal{H}_+$ is the homeomorphism described in Theorem 13.

References

[BR1] Bratteli, O., Robinson D. W.: *Operator Algebras and Quantum Statistical Mechanics 1*. Second edition, Springer, Berlin (2002).

[RVD] Rieffel, M.A., van Daele, A.: A bounded operator approach to Tomita-Takesaki theory. Pacific J. Math. **69**, 187 (1977).

[SZ] Stratila, S., Zsido, L.: *Lectures on von Neumann Algebras.* Abacus Press, Tunbridge Wells (1979).

[T] Takesaki, M.: *Tomita's Theory of Modular Hilbert Algebras and its Applications.* Lecture Notes in Mathematics **128**. Springer, Berlin, 1970.