Entropy production in repeated quantum measurements

Joint work with Tristan Benoist (Toulouse), Vojkan Jakšić (McGill)
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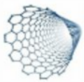

Irreversibility vs Measurements

- 1927: Heisenberg *”reduction of the wave function”*
- 1927: Eddington *”time”s arrow”*
- 1932: von Neumann *”quantum arrow of time”*
- 1937: Landau-Lifschitz *”quantum vs thermodynamic arrow of time”*
- . . .
- 1963; Wigner *”causal vs statistical evolution”*
- . . .
- 1991: Zurek *”environment* $\Rightarrow$ *decoherence”*
- . . .

Irreversibility vs Measurements

- 1927: Heisenberg *"reduction of the wave function"* is the result of the *"dephasing"* due to interactions with the measuring apparatus:

> Dies hat zur Folge, dass die endgültige Transformationsmatrix e_{nl} [...] nicht mehr durch $\sum_m c_{nm} d_{ml}$ gegeben ist, sondern jedes Glied der Summe hat noch einen unbekannten Phasenfaktor. Wir können also nur erwarten, dass der Mittelwert von $e_{nl}\bar{e}_{nl}$ über alle diese eventuellen Phasenänderungen gleich Z_{nl} ist. Eine einfache Rechnung ergibt, dass dies der Fall ist.
>
> *Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik.* Z. Phys. **43**, 172–198 (1927).

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Irreversibility vs Measurements

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> If as we follow the arrow we find more and more of the random element in the state of the world, then the arrow is pointed towards the future; if the random element decreases, the arrow points towards the past.
>
> *The Nature of the Physical World.* McMillan, London, 1928.

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> Therefore, we have reached a point at which it is desirable to utilize the thermodynamical method of analysis, because it alone makes it possible for us to understand correctly the difference between **1.** [reduction] and **2.** [unitary evolution], into which reversibility questions obviously enter.
>
> *Mathematical Foundations of Quantum Mechanics.* Princeton University Press, 1955.

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  > Thus in quantum mechanics there is a physical non-equivalence of the two directions of time, and theoretically the law of increase of entropy might be its macroscopic expression.
  >
  > *Statistical Physics.* Pergamon, 1978.

- ...
- 1963; Wigner *"causal vs statistical evolution"*
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Irreversibility vs Measurements

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> Returning to the problem of measurement, we see that we have not arrived either at a conflict between the theory of measurement and the equations of motion, nor have we obtained an explanation of that theory in terms of the equations of motion [. . .] However, the fundamental point remains unchanged and a full description of an observation must remain impossible since the quantum-mechanical equations of motion are causal and contain no statistical element, whereas the measurement does.
>
> *The problem of measurements.* Amer. J. Phys. **31**, 6–15 (1963).

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> ...recent years have seen a growing consensus that progress is being made in dealing with the measurement problem. The key (and uncontroversial) fact has been known almost since the inception of quantum theory, but its significance for the transition from quantum to classical is being recognized only now: Macroscopic systems are never isolated from their environments. [...] The resulting "decoherence" can not be ignored when one addresses the problem of the reduction of wavepackets...
>
> *Decoherence and the transition from quantum to classical.* Physics Today, October 1991, 36–44.

Irreversibility vs Measurements

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- ...
- More recently: *"statistical mechanics of repeated measurements"*
  Kümmerer-Maassen'04, Barchielli-Gregoratti'09,
  Bauer-Benoist-Bernard'11, Benoist-Pellegrini'14,
  Ballesteros-Fraas-Fröhlich-Schubnel'16,...

Fluctuation Relations: “Microscopic” form of the 2$^{\mathrm{nd}}$ Law

Classical

- Evans-Cohen-Morriss: Probability of second law violation in shearing steady flows. Phys. Rev. Lett. **71**, 2401 (1993).
- Gallavotti-Cohen: Dynamical ensembles in nonequilibrium statistical mechanics. Phys. Rev. Lett. **74**, 2694 (1995).
- . . .
- Ciliberto-Garnier-Hernandez-Lacpatia-Pinton-Ruiz Chavarria: Experimental test of the Gallavotti–Cohen fluctuation theorem in turbulent flows. Physica A **340** 240 (2004).
- . . .
- Ciliberto-Imparato-Naert-Tana: Heat flux and entropy produced by thermal fluctuations. Phys. Rev. Lett. **110**, 180601 (2013).
- . . .

Fluctuation Relations: “Microscopic” form of the 2^{nd} Law

Quantum

- Kurchan & Tasaki (2000): Extension to quantum dynamics.
- Andrieux-Gaspard-Monnai-Tasaki: The fluctuation theorem for currents in open quantum systems. New J. Phys. **11** 043014 (2009).
- . . .
- Jakšić-Ogata-P-Seiringer: Quantum hypothesis testing and non-equilibrium statistical mechanics. Rev. Math. Phys. **24**, 1230002 (2012).
- Batalhão-Souza-Sarthour-Oliveira-Paternostro-Lutz-Serra: Irreversibility and the arrow of time in a quenched quantum system. Phys. Rev. Lett. **115**, 190601 (2015).

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In this talk:

- Emergence of the ”arrow of time” in repeated quantum measurement processes
- Relation with the Gallavotti-Cohen ”fluctuations relations”
- Thermodynamic formalism for entropy production in classical dynamical systems

Framework

(Orthodox) Repeated Quantum Measurements

- Finite dimensional Hilbert space $\mathcal{H}$
- Finite alphabet $\mathcal{A} = \{1, 2, \ldots, \ell\}$
- Quantum instrument $\{\Phi_a\}_{a\in\mathcal{A}}$
  - CP maps $\Phi_a : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$
  - Unital $\Phi(\mathbb{1}) = \sum_{a\in\mathcal{A}} \Phi_a(\mathbb{1}) = \mathbb{1}$
  - Duality $\mathrm{tr}(\Phi_a^*(X)Y) = \mathrm{tr}(X\Phi_a(Y))$
- Initial state ρ
- Probability measures on finite "quantum trajectories"

$$\mathbb{P}_T(\omega_1\omega_2\cdots\omega_T) = \mathrm{tr}(\rho\,\Phi_{\omega_1}\circ\Phi_{\omega_2}\circ\cdots\Phi_{\omega_T}(\mathbb{1}))$$

(Lüders-Schwinger-Wigner formula) extend to a probability $\mathbb{P}$ on $\Omega = \mathcal{A}^{\mathbb{N}}$ as a consequence of unitality (Kolmogorov theorem)

$$\sum_{\omega_{T+1},\ldots,\omega_{T+S}} \mathbb{P}_{T+S}(\omega_1,\ldots,\omega_T,\omega_{T+1},\ldots,\omega_{T+S}) = \mathbb{P}_T(\omega_1,\ldots,\omega_T)$$

- Time-reversal

$$\Theta_T(\omega_1\omega_2\cdots\omega_T) = \theta(\omega_T)\cdots\theta(\omega_2)\theta(\omega_1)$$

for some involution $\theta : \mathcal{A} \to \mathcal{A}$ (e.g., spin flip)

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Assumption A

Initial state ρ is faithful and invariant: $\rho > 0$, $\Phi^*(\rho) = \rho$

Basic Properties

- $\Phi^*(\rho) = \rho \Rightarrow \mathbb{P}$ is invariant under the left shift $\tau : (\omega_1, \omega_2, \ldots) \mapsto (\omega_2, \omega_3, \ldots)$

  Quantum instrument $(\{\Phi_a\}_{a\in\mathcal{A}}, \rho) \Longrightarrow$ classical dynamical system $(\Omega, \tau, \mathbb{P})$

  From the quantum mechanical perspective

$$\sum_{\omega_1,\ldots,\omega_T} \mathbb{P}_{T+S}(\omega_1, \ldots, \omega_T, \omega_{T+1}, \ldots, \omega_{T+S}) = \mathbb{P}_S(\omega_{T+1}, \ldots, \omega_{T+S})$$

  is a **decoherence** assumption.

- $\rho > 0 \Rightarrow$ the upper quasi-Bernoulli property holds

$$\mathbb{P}_{T+T'} \leq C\, \mathbb{P}_T \mathbb{P}_{T'} \circ \tau^{-T}, \qquad \widehat{\mathbb{P}}_{T+T'} \leq C\, \widehat{\mathbb{P}}_T \widehat{\mathbb{P}}_{T'} \circ \tau^{-T}$$

- The probability of time-reversed trajectories $\widehat{\mathbb{P}}_T = \mathbb{P}_T \circ \Theta_T$ describes the instrument $\{\widehat{\Phi}_a\}_{a\in\mathcal{A}}$ ([Crooks'08])

$$\widehat{\Phi}_a(X) = \rho^{-1/2} \Phi^*_{\theta(a)}(\rho^{1/2} X \rho^{1/2}) \rho^{-1/2}$$

- If 1 is simple eigenvalue of Φ, then $\mathbb{P}$ is ergodic ($\Leftarrow \Phi$ irreducible)

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Remark

Special case of ”finitely correlated states” or ”matrix product states” of 1D spin chains [Fannes-Nachtergaele-Werner'92]

Basic Properties

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Goal

Quantify the emergence of the arrow of time as a "distance" between $\mathbb{P}_T$ and $\widehat{\mathbb{P}}_T$ in the limit $T \to \infty$

Strategy

- Universal mechanism for entropic fluctuation relations out of equilibrium
- Applies to classical and quantum dynamical systems
- Need to develop a thermodynamic formalism for non-Gibbsian dynamical systems
- Motivated by a body of recent works on subadditive ergodic theory and multifractal analysis of measures [Feng-Käenmäki-Barreira,...]

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Assumption B

$\mathbb{P}$ is ergodic and $\mathrm{supp}\,\widehat{\mathbb{P}}_T = \mathrm{supp}\,\mathbb{P}_T$ for all T (large enough)

- $\mathbb{P}_T$ and $\widehat{\mathbb{P}}_T$ equivalent for all T (large enough)
- Entropy production reflects the dichotomy:

$$\mathbb{P} = \widehat{\mathbb{P}} \text{ (equilibrium, detailed balance) or } \mathbb{P} \perp \widehat{\mathbb{P}} \text{ (nonequilibrium)}$$

- Out of equilibrium, the separation between $\mathbb{P}_T$ and $\widehat{\mathbb{P}}_T$ as $T \to \infty$ is quantified by relative entropies

$$S(\mathbb{P}_T|\widehat{\mathbb{P}}_T) = \mathbb{P}_T(\sigma_T) \geq 0, \qquad S_\alpha(\mathbb{P}_T|\widehat{\mathbb{P}}_T) = \log \mathbb{P}_T(\mathrm{e}^{-\alpha\sigma_T})$$

expectation and cumulant generating function of the entropy production random variable

$$\sigma_T(\omega) = \log \frac{\mathbb{P}_T(\omega)}{\widehat{\mathbb{P}}_T(\omega)} = -\sigma_T \circ \Theta_T(\omega)$$

Fluctuation relations

$$P_T(s) = \mathbb{P}\left(\left\{\omega \mid \frac{1}{T}\sigma_T(\omega) = s\right\}\right)$$

- Law of mean entropy production rate on $[0, T]$
- Assumption B $\Rightarrow P_T(s) > 0 \Leftrightarrow P_T(-s) > 0$
- Relative entropies

$$S(\mathbb{P}_T|\widehat{\mathbb{P}}_T) = \sum sP_T(s) \geq 0, \qquad S_\alpha(\mathbb{P}_T|\widehat{\mathbb{P}}_T) = \log \sum \mathrm{e}^{-\alpha s} P_T(s)$$

- Symmetry of the Rényi entropy $S_{1-\alpha}(\mathbb{P}_T|\widehat{\mathbb{P}}_T) = S_\alpha(\mathbb{P}_T|\widehat{\mathbb{P}}_T)$ yields the finite-time fluctuation relation

$$\frac{P_T(-s)}{P_T(s)} = \mathrm{e}^{-Ts}$$

- More (LDP, CLT, Chernoff & Hoeffding exponents, Gallavotti-Cohen fluctuation relations) if we can control

$$\lim_{T\to\infty} \frac{1}{T} S_\alpha(\mathbb{P}_T|\widehat{\mathbb{P}}_T)$$

Entropy production

Results from ergodic theory

- Gibbs-Shannon entropy: $S(\mathbb{P}_T) = -\sum_{\omega\in\mathcal{A}^T} \mathbb{P}_T(\omega)\log\mathbb{P}_T(\omega)$
- Kolmogorov-Sinai entropy: $S(\mathbb{P}) = \lim_{T\to\infty} T^{-1}S(\mathbb{P}_T) \in [0,\log\ell]$
- Shannon-McMillan-Breiman: $S(\mathbb{P}) = -\lim_{T\to\infty} T^{-1}\log\mathbb{P}_T(\omega_1,\ldots,\omega_T)$, for $\mathbb{P}$-a.e. ω and in $L^1(\Omega,\mathbb{P})$
- Gibbs property (Bowen)

$$C^{-1}\mathrm{e}^{-\sum_{t=1}^{T}\varphi\circ\tau^t} \le \mathbb{P}_T \le C\,\mathrm{e}^{-\sum_{t=1}^{T}\varphi\circ\tau^t}$$

  for some (Hölder) continuous potential φ (Gallavotti-Cohen *chaotic hypothesis*) generally fails for repeated measurement processes $\Rightarrow$ need thermodynamic formalism for non-Gibbsian systems
- Weaker than Gibbs: Upper & Lower Quasi-Bernoulli properties

$$C^{-1}\mathbb{P}_T\,\mathbb{P}_{T'}\circ\tau^T \le \mathbb{P}_{T+T'} \le C\,\mathbb{P}_T\,\mathbb{P}_{T'}\circ\tau^T$$

  implies existence and differentiability of the entropic pressure

$$\mathbb{R}\ni\alpha\mapsto e(\alpha) = \lim_{T\to\infty}\frac{1}{T}S_\alpha(\mathbb{P}_T|\widehat{\mathbb{P}}_T)$$

- Dynamical analogue of thermodynamic free energy/pressure
- Assumptions A & B only ensure upper quasi-Bernoulli $\Rightarrow$ $e(\alpha)$ may develop singularities: dynamical phase transition

Entropy production: Level I

Theorem I (Entropy production)

Under Assumptions A & B

- Mean entropy production rate

$$\lim_{T\to\infty} \frac{1}{T}\mathbb{P}_T(\sigma_T) = \mathrm{Ep} \geq 0$$

- Strong law of large numbers: $\mathbb{P}$-a.s.

$$\lim_{T\to\infty} \frac{1}{T}\sigma_T(\omega) = \mathrm{Ep} \tag{1}$$

- If $\mathrm{Ep} < \infty$ then (1) holds in $L^1(\Omega, \mathbb{P})$
- Stein's exponent: Let $s_T(\epsilon) = \min\{\widehat{\mathbb{P}}_T(A) \,|\, A \in \mathcal{A}^T, \mathbb{P}_T(A) \geq \epsilon\}$ for $\epsilon \in]0,1[$

$$\lim_{T\to\infty} \frac{1}{T}\log s_T(\epsilon) = -\mathrm{Ep}$$

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Message

- $\mathrm{Ep} = 0 \Leftrightarrow \widehat{\mathbb{P}} = \mathbb{P}$ & $\mathrm{Ep} > 0 \Leftrightarrow \widehat{\mathbb{P}} \perp \mathbb{P}$
- $\mathbb{P}_T(A_T) \geq \epsilon > 0$ for large $T \Rightarrow \widehat{\mathbb{P}}_T(A_T) \lesssim \mathrm{e}^{-T\mathrm{Ep}}$ exponential separation of the supports of $\mathbb{P}_T$ and $\widehat{\mathbb{P}}_T$

Entropy production: Level II

Rényi's relative entropy

$$e_T(\alpha) = S_\alpha(\mathbb{P}_T|\widehat{\mathbb{P}}_T) = \log \mathbb{P}_T(\mathrm{e}^{-\alpha\sigma_T})$$

- is a convex function of α
- has left/right derivatives $\partial^\pm e(\alpha)$ where finite
- is non-positive for $\alpha \in [0, 1]$
- is non-negative for $\alpha \in \mathbb{R} \setminus [0, 1]$
- vanishes at $\alpha = 0$ and $\alpha = 1$
- satisfies $e_T(1-\alpha) = e_T(\alpha)$ (the *Gallavotti-Cohen symmetry*)

All these properties will be preserved in the limit (= entropic pressure)

$$e(\alpha) = \lim_{T\to\infty} \frac{1}{T} e_T(\alpha)$$

whenever it exists.

Entropy production: Level II

Theorem II (thermodynamic formalism for $\alpha \in [0,1]$)

Suppose that Assumptions A & B hold and denote by $\mathcal{P}_\tau$ the set of τ-invariant probability measures on Ω.

- The entropic pressure $e(\alpha)$ exists for all $\alpha \in [0,1]$.
- Either $e(\alpha) = -\infty$ for all $\alpha \in]0,1[$, or $e(\alpha) > -\infty$ for all $\alpha \in]0,1[$.
- The limit

$$f(\mathbb{Q}) = \lim_{T\to\infty} \frac{1}{T}\mathbb{Q}(\log \mathbb{P}_T - \log \mathbb{Q}_T)$$

  exists for all $\mathbb{Q} \in \mathcal{P}_\phi$ and satisfies

$$\alpha f(\mathbb{Q}) + (1-\alpha) f(\widehat{\mathbb{Q}}) \geq \limsup_{T\to\infty} \frac{1}{T} e_T(\alpha)$$

  for all $\alpha \in \mathbb{R}$.
- For $\alpha \in [0,1]$

$$e(\alpha) = \sup_{\mathbb{Q}\in\mathcal{P}_\tau} \alpha f(\mathbb{Q}) + (1-\alpha) f(\widehat{\mathbb{Q}})$$

Entropy production: Level II

Theorem II (thermodynamic formalism, cont'd)

- If $e(\alpha)$ is finite for $\alpha \in [0,1]$ then

$$\mathcal{P}_{\rm eq}(\alpha) = \{\mathbb{Q} \in \mathcal{P}_\tau \,|\, \alpha f(\mathbb{Q}) + (1-\alpha) f(\widehat{\mathbb{Q}}) = e(\alpha)\}$$

  is a non-empty compact convex subset of $\mathcal{P}_\tau$, a Choquet simplex and a face of $\mathcal{P}_\tau$ whose extreme points are ergodic.

- For $\alpha \in]0,1[$

$$\partial^- e(\alpha) = \inf_{\mathbb{Q} \in \mathcal{P}_{\rm eq}(\alpha)} f(\widehat{\mathbb{Q}}) - f(\mathbb{Q}) \le \sup_{\mathbb{Q} \in \mathcal{P}_{\rm eq}(\alpha)} f(\widehat{\mathbb{Q}}) - f(\mathbb{Q}) = \partial^+ e(\alpha)$$

- $\mathcal{P}_{\rm eq}(0) = \{\mathbb{P}\}$, $\mathcal{P}_{\rm eq}(1) = \{\widehat{\mathbb{P}}\}$ and

$$-\partial^+ e(0) = \mathrm{Ep} = \partial^- e(1)$$

Remark

- If $f(\mathbb{Q})$ and $f(\widehat{\mathbb{Q}})$ are finite, then $f(\widehat{\mathbb{Q}}) - f(\mathbb{Q}) = -\lim_{T\to\infty} \frac{1}{T} \mathbb{Q}(\sigma_T)$

Entropy production: Level II

Assumption C (weaker than lower quasi-Bernoulli)

There exists $T^* > 0$ and $C_{T^*} > 0$ such that

$$\max_{|\xi| \leq T^*} \frac{\mathbb{P}(\omega\xi\nu)\widehat{\mathbb{P}}(\omega\xi\nu)}{\mathbb{P}(\omega)\mathbb{P}(\nu)\widehat{\mathbb{P}}(\omega)\widehat{\mathbb{P}}(\nu)} \geq C_{T^*}$$

for all finite words ω, ν (i.e., cylinder sets)

Remarks

- Minimal assumption for Theorem III
- Often easy to check in concrete models
- Irreducibility of $\sum_a \Phi_a \otimes \widehat{\Phi}_a \Rightarrow$ C
- Simple algebraic criterion in terms of Kraus representations

Entropy production: Level II

Theorem III (Differentiability on $]0,1[$)

Under Assumptions A,B & C

- $\alpha \in [0,1] \Rightarrow \mathcal{P}_{\rm eq}(\alpha)$ is a singleton: $e(\alpha)$ is differentiable on $]0,1[$.
- For any open set $O \subset]-\mathrm{Ep}, \mathrm{Ep}[$ the local LDP

$$\lim_{T\to\infty} \frac{1}{T}\mathbb{P}\left(\left\{\omega \in \Omega \,\middle|\, \frac{1}{T}\sigma_T(\omega) \in O\right\}\right) = -\inf_{s\in O} I(s),$$

holds with rate function $I(s) = -\inf_{\alpha\in[0,1]}(\alpha s + e(\alpha))$ satisfying the fluctuation relation

$$I(-s) = I(s) + s$$

- Chernoff exponent: $\lim_{T\to\infty} \frac{1}{T}\log(2 - \|\mathbb{P}_T - \widehat{\mathbb{P}}_T\|_{\rm var}) = e(1/2)$
- Hoeffding exponent: for $s \geq 0$

$$\inf_{\{A_T\subset\mathcal{A}^T\}}\left\{\limsup_T \frac{1}{T}\log\widehat{\mathbb{P}}_T(A_T) \,\middle|\, \limsup_T \frac{1}{T}\log\mathbb{P}_T(\mathcal{A}^T\setminus A_T) < -s\right\}$$

$$= -\sup_{\alpha\in[0,1]} \frac{-s\alpha - e(\alpha)}{1-\alpha}$$

Entropy production: Level II

Remarks

- Assuming the lower Quasi-Bernoulli property one can show that $e(\alpha)$ is differentiable on $\mathbb{R}$. As a consequence, the LDP for the mean entropy production rate holds for all open sets $O \subset \mathbb{R}$.
- Φ_a positivity improving for all $a \in \mathcal{A} \Rightarrow$ lower quasi-Bernoulli ($\sim$ Gallavotti-Cohen chaotic hypothesis)
- We have simple examples of repeated measurement processes for which Assumptions A, B & C hold but the lower quasi-Bernoulli property fails in a strong way
$$\frac{\mathbb{P}_T(\omega_T \nu_T)}{\mathbb{P}_T(\omega_T)\mathbb{P}_T(\nu_T)} \sim \mathrm{e}^{-\gamma T}, \qquad \gamma > 0$$
Nevertheless, in these example the entropic pressure $e(\alpha)$ exist and is finite for all $\alpha \in \mathbb{R}$. It exhibits a second order phase transition at $\alpha = 0/1$.

Examples

A Markov instrument: $\Phi_{(i,j)}(X) = p_{ij}|i\rangle\langle j|X|j\rangle\langle i|$

- $p = (p_{ij})$ stochastic matrix $p_{ij} > 0 \Rightarrow$ unique invariant state $\pi p = \pi$
- Time-reversal $\theta(i,j) = (j,i)$
- $\{|i\rangle\}$ ON-basis: A holds with $\rho = \sum_i \pi_i |i\rangle\langle i|$
- B holds and $\mathrm{Ep} = \sum_{i,j} \pi_i p_{ij} \log \frac{p_{ij}}{p_{ji}}$
- $\mathrm{Ep} = 0$ iff detailed balance $\pi_i p_{ij} = \pi_j p_{ji}$ holds
- C holds. Entropic pressure $e(\alpha)$ is given by the spectral radius of the matrix

$$m(\alpha) = (p_{ij}^{1-\alpha} p_{ji}^{\alpha})$$

- Lower quasi-Bernoulli fails, nevertheless $\mathbb{R} \ni \alpha \mapsto e(\alpha)$ is real analytic

Examples

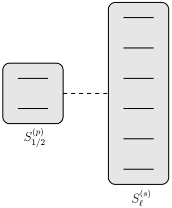
A Bernoulli instrument (ancilla measurement of $S_{1/2}$ in $S_\ell \otimes S_{1/2}$)

Let $\vec{S}^{(s)}$ and $\vec{S}^{(p)}$ denote spin ℓ and spin 1/2 operators, $\epsilon, \omega, \lambda, t \in \mathbb{R}$, $\eta \in]0,1[$

$$H = \epsilon S_3^{(p)} + \omega S_3^{(s)} + \lambda \vec{S}^{(p)} \cdot \vec{S}^{(s)}$$

$$\rho^{(p)} = \frac{1}{2} + (2\eta - 1) S_3^{(p)}, \qquad P_\pm^{(p)} = \frac{1}{2} \pm S_3^{(p)}$$

$$\Phi_\pm^*(\rho) = (\mathrm{Id}^{(s)} \otimes \mathrm{tr}^{(p)})((\mathbb{1} \otimes P_\pm^{(p)}) \mathrm{e}^{-itH} (\rho \otimes \rho^{(p)}) e^{itH}), \quad \theta(\pm) = \mp$$

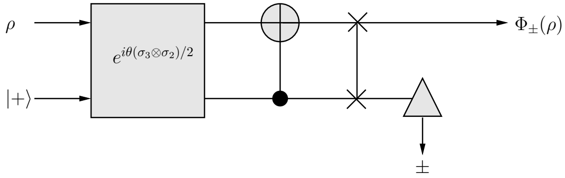


Assumptions A,B & C hold. $\mathbb{P}$ is Bernoulli,

$$\mathrm{Ep} = (2\eta - 1) \log \frac{\eta}{1 - \eta}, \qquad e(\alpha) = \log(\eta^\alpha (1-\eta)^{1-\alpha} + \eta^{1-\alpha}(1-\eta)^\alpha)$$

A quasi-Bernoulli perfect Kraus instrument: $\Phi_\pm(X) = V_\pm X V_\pm^*$

$$V_- = \begin{pmatrix} 0 & -\sin\theta/2 \\ \cos\theta/2 & 0 \end{pmatrix}, \quad V_+ = \begin{pmatrix} \cos\theta/2 & 0 \\ 0 & \sin\theta/2 \end{pmatrix}, \quad \theta(\pm) = \mp$$



- Satisfies Assumptions A,B & C for $\theta \in]0, \pi/2[$.
- $\mathbb{P}$ is quasi-Bernoulli but not Bernoulli.
- Entropic pressure is real analytic.

Examples

A Non quasi-Bernoulli perfect Kraus instrument: $\Phi_\pm(X) = V_\pm X V_\pm^*$

$$V_- = \begin{pmatrix} \sqrt{\cos\theta} & -\sin\theta/2 \\ -\sin\theta/2 & 0 \end{pmatrix}, \quad V_+ = \begin{pmatrix} -\sin\theta/2 & 0 \\ -\sqrt{\cos\theta} & -\sin\theta/2 \end{pmatrix}, \quad \theta(\pm) = \mp$$

Satisfies Assumptions A, B & C for $\theta \in]0, \pi/2[$ but is not Lower Quasi-Bernoulli

$$\lim_{T\to\infty} \frac{1}{2T+1} \log \frac{\mathbb{P}_{2T+1}(-\cdots-+-\cdots-)}{\mathbb{P}_{T+1}(-\cdots-+)\mathbb{P}_T(-\cdots-)} = -\xi = -\log \frac{1+\sqrt{1-4\sin^4\theta/2}}{2\sin^2\theta/2} < 0$$

Examples

A Non quasi-Bernoulli perfect Kraus instrument: $\Phi_{\pm}(X) = V_{\pm} X V_{\pm}^{*}$

$$V_{-} = \begin{pmatrix} \sqrt{\cos\theta} & -\sin\theta/2 \\ -\sin\theta/2 & 0 \end{pmatrix}, \quad V_{+} = \begin{pmatrix} -\sin\theta/2 & 0 \\ -\sqrt{\cos\theta} & -\sin\theta/2 \end{pmatrix}, \quad \theta(\pm) = \mp$$

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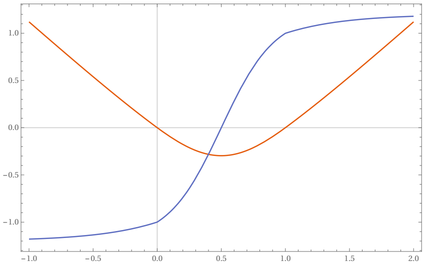

Entropic pressure and its derivative for $\theta = \pi/3$

Examples

A Non quasi-Bernoulli perfect Kraus instrument: $\Phi_\pm(X) = V_\pm X V_\pm^*$

$$V_- = \begin{pmatrix} \sqrt{\cos\theta} & -\sin\theta/2 \\ -\sin\theta/2 & 0 \end{pmatrix}, \quad V_+ = \begin{pmatrix} -\sin\theta/2 & 0 \\ -\sqrt{\cos\theta} & -\sin\theta/2 \end{pmatrix}, \quad \theta(\pm) = \mp$$

Satisfies Assumptions A, B & C for $\theta \in]0, \pi/2[$ but is not Lower Quasi-Bernoulli

$$\lim_{T\to\infty} \frac{1}{2T+1} \log \frac{\mathbb{P}_{2T+1}(-\cdots - + - \cdots -)}{\mathbb{P}_{T+1}(-\cdots - +)\mathbb{P}_T(-\cdots -)} = -\xi = -\log \frac{1 + \sqrt{1 - 4\sin^4\theta/2}}{2\sin^2\theta/2} < 0$$

Central limit theorem fails: as $T \to \infty$

$$\frac{\sigma_T - \mathbb{P}(\sigma_T)}{\sqrt{T}} \Rightarrow \frac{\xi}{\cosh\xi}(u - |v|)$$

with $u, v \sim \mathcal{N}(0,1)$.

Perspectives

- Further develop the thermodynamic formalism for non-Gibbsian systems using results from the subadditive ergodic theory.
- Criteria for analyticity, occurrence of first order phase transitions ?
- Investigate the physical meaning of phase transition beyond the failure of CLT. Occurrence of anomalous scaling ?
- Special measurements, e.g., thermal probes.
- Continuous measurements/monitoring.
- . . .

Thank you !