Markovian Repeated Interaction Systems

A joint work with Jean-François Bougron and Alain Joye

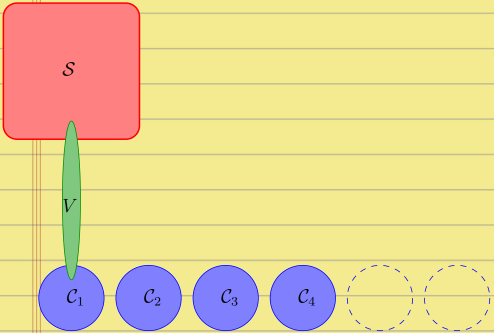
Claude-Alain Pillet

Quantum Trajectories

Toulouse, October 18–22 2021

Introduction

Repeated Interaction Quantum Systems



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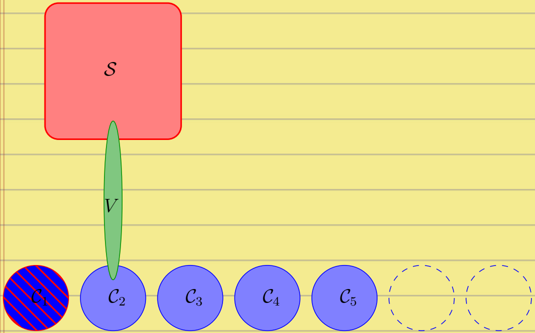

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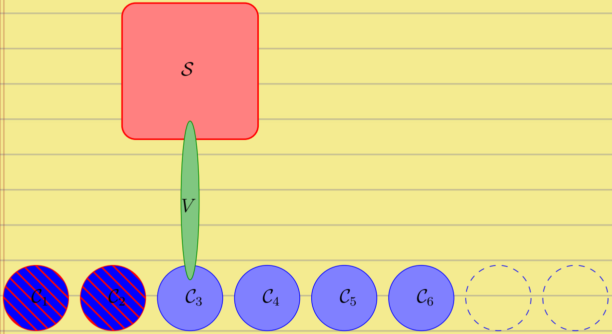

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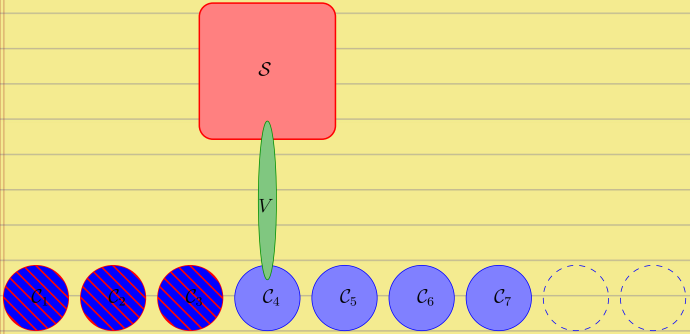

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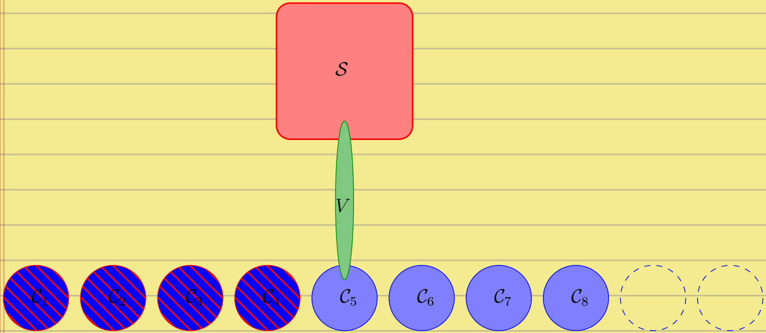

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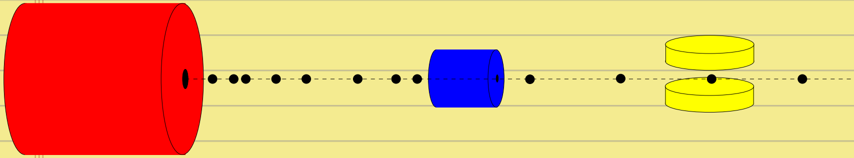


Background I: Repeated Interactions

⚠ "Quantum Markov" has been used and abused !

- Quantum Markov chains: Wellens, Buchleitner, Kümmerer, Maassen (2000)
- Continuous limit: Attal, Pautrat (2006)
- Weak coupling: Attal, Joye (2007)
- Long time asymptotics: Bruneau, Joye, Merkli (2006–2020)
- Non-Markovian: Pellegrini, Petruccione (2009)
- Metastability: Bruneau, P. (2009)
- Random: Nechita, Pellegrini (2012), Bougron, Bruneau (2020)
- Parameter estimation: Guţă, Kiukas (2015)
- CLT, large deviations: Guţă, van Horssen (2015)
- . . . .

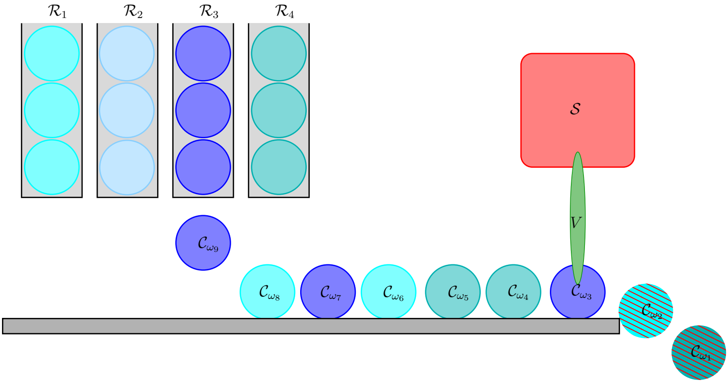
Background II: Repeated Measurements . . .

. . . and Quantum Trajectories

- Barchielli, Belavkin (1991)
- Kümmerer, Maassen (2000–)
- Pellegrini (2009–)
- Bauer, Benoist, Bernard, Tilloy (2011–)
- Haroche et al., Wineland et al. (Nobel 2012)
- Benoist, Cuneo, Jakšić, Pautrat, P. (2018–)
- Benoist, Fraas, Pautrat, Pellegrini (2019)
- Ballesteros, Benoist, Crawford, Fraas, Fröhlich, Schubnel (2020–)
- . . . .

Setup

System–Reservoirs Picture



Abstract Setup

The Driving Classical Markov Chain

- Finite state space Ω.
- Transition matrix $P \in \mathbb{R}^{\Omega \times \Omega}$ (right stochastic).
- Probability vector $\pi \in \mathbb{R}^{\Omega}$.
- Markov chain $\boldsymbol{\omega} = (\omega_n)_{n \in \mathbb{N}} \in \Omega^{\mathbb{N}}$ with probability measure

$$\mathbb{P}(\omega_0 = i_0, \ldots, \omega_k = i_k) = \pi_{i_0} P_{i_0 i_1} \cdots P_{i_{k-1} i_k}.$$

- Left shift on $\Omega^{\mathbb{N}}$: $\boldsymbol{\omega} = (\omega_0, \omega_1, \ldots) \mapsto \sigma(\boldsymbol{\omega}) = (\omega_1, \omega_2, \ldots)$.

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The Quantum Machinery

System $\mathcal{S}$ with finite dimensional C^*-algebra of observables $\mathcal{A}$.

- $(\rho_\omega)_{\omega\in\Omega}$ family of states on $\mathcal{A}$.
- $(\mathcal{L}_\omega)_{\omega\in\Omega}$ family of CPTP maps on $\mathcal{A}_*$.

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The Random Dynamical System

$$\rho_n(\boldsymbol{\omega}) = \mathcal{L}_{\omega_n} \cdots \mathcal{L}_{\omega_1} \rho_{\omega_0}.$$

is a Markovian Repeated Interaction System (MRIS)

A Feynman–Kac Formalism

- C^*-algebra
$$\mathfrak{A} = \mathcal{A}^\Omega = \{\Omega \ni \omega \mapsto X(\omega) \in \mathcal{A}\}$$
diagonal subalgebra of the full algebra of $\Omega \times \Omega$ matrices with entries in $\mathcal{A}$ equipped with the trace
$$\mathrm{Tr}\, X = \sum_{\omega\in\Omega} \mathrm{tr} X(\omega).$$
Extended observables $X \in \mathfrak{A}$ depend on which reservoir $\mathcal{S}$ is interacting with.
- $\mathfrak{A}_* = \mathcal{A}_*^\Omega$ with duality
$$\mathfrak{A}_* \times \mathfrak{A} \ni (R, X) \mapsto \langle R, X\rangle = \sum_{\omega\in\Omega} \langle R(\omega), X(\omega)\rangle$$
- Extended states $R \in \mathfrak{A}_*$, $R(\omega) \geq 0$, $\mathrm{Tr}\, R = 1$.
- Feynman–Kac CPTP map on $\mathfrak{A}_*$
$$(\mathbb{L}R)(\omega) = \sum_{\nu\in\Omega} P_{\nu\omega} \mathcal{L}_\nu R(\nu)$$

The Semigroup Structure

$$R_n(\omega) = \mathbb{E}[\rho_n(\boldsymbol{\omega}) 1_{\omega_{n+1}=\omega}] = \mathbb{E}[\rho_n(\boldsymbol{\omega}) | \omega_{n+1} = \omega] \mathbb{P}(\omega_{n+1} = \omega)$$

Lemma 1

- $\operatorname{tr} R_n(\omega) = \mathbb{P}(\omega_{n+1} = \omega) = (\pi P^{n+1})_\omega = \pi_\omega^{(n+1)}$
- $\displaystyle\sum_{\omega \in \Omega} R_n(\omega) = \mathbb{E}[\rho_n(\boldsymbol{\omega})]$
- $R_n(\omega) = (\mathbb{L}^n R_0)(\omega)$
- $\mathbb{E}[\langle \rho_n(\boldsymbol{\omega}), X(\omega_{n+1}\rangle] = \langle \mathbb{L}^n R_0, X \rangle$

Proof. By elementary direct calculation.

Extended Steady States

Definition

An extended state R_+ such that $\mathbb{L}R_+ = R_+$ is called Extended Steady State (ESS).

- R_+ always exists, but may not be unique.

$$\pi_{+\omega} = \mathrm{tr} R_+(\omega), \qquad \rho_{+\omega} = \frac{\mathcal{L}_\omega R_+(\omega)}{\pi_{+\omega}},$$

- $\pi_+ P = \pi_+$: the Markov chain started with π_+ is stationary.
- The repeated interaction process driven by this stationary chain and started with ρ_+ is stationary

$$\mathbb{E}_+[\langle \rho_{+n}(\boldsymbol{\omega}), X(\omega_{n+1}\rangle] = \langle R_+, X\rangle$$

- When unique, R_+ convey important information on the large time asymptotics of the process.

Primitivity, Irreducibility

Definition

- $\mathbb{L}$ is positivity improving if, for any $R \in \mathfrak{A}_*$, $R \geq 0 \implies \mathbb{L}R > 0$.
- $\mathbb{L}$ is primitive if $\mathbb{L}^n$ is positivity improving for some $n > 0$.
- $\mathbb{L}$ is irreducible if $\mathrm{e}^{t\mathbb{L}}$ is positivity improving for some $t > 0$.

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- $\mathrm{sp}(\mathbb{L})$ is a subset of the closed unit disk containing 1, and the eigenspace associated to the eigenvalue 1 contains an extended steady state R_+.
- $\mathbb{L}$ is irreducible iff its eigenvalue 1 is simple. In this case, there is a unique extended steady state R_+. Moreover, R_+ is faithful and for any $R \in \mathfrak{A}_*$

$$\left| \frac{1}{n} \sum_{k=0}^{n-1} \left(\mathbb{L}^k R - \langle R, \mathbb{1} \rangle R_+ \right) \right| = \|R\| \mathcal{O}(n^{-1})$$

as $n \to \infty$.

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  as $n \to \infty$.
- $\mathbb{L}$ is primitive iff its simple eigenvalue 1 is gapped

  $$\Delta = \max\{|z| \mid z \in \mathrm{sp}(\mathbb{L}) \setminus \{1\}\} < 1.$$

  Moreover, for any $R \in \mathfrak{A}_*$ and $\epsilon > 0$,

  $$\left| \mathbb{L}^n R - \langle R, \mathbb{1} \rangle R_+ \right| = \|R\| \mathcal{O}((\Delta + \epsilon)^n)$$

Primitivity, Irreducibility

Lemma 2

Set

$$\bar{\mathcal{L}} = \sum_{\omega \in \Omega} \pi_\omega \mathcal{L}_\omega$$

where π is a faithful probability vector, and note that $\bar{\mathcal{L}}$ is a CPTP map on $\mathcal{A}_*$.

- $\mathbb{L}$ is positivity improving iff P and every $\mathcal{L}_\omega$ are.
- If $\mathbb{L}$ is irreducible (resp. primitive), so are P and $\bar{\mathcal{L}}$.
- If P is positivity improving, then $\mathbb{L}$ is irreducible (resp. primitive) iff $\bar{\mathcal{L}}$ is.
- If every $\mathcal{L}_\omega$ is positivity improving, then $\mathbb{L}$ is irreducible (resp. primitive) iff P is.

Ergodic Theory of MRIS

A Random Ergodic Theorem

[Beck–Schwartz 1957]

Let $\mathfrak{X}$ be a reflexive Banach space and (S, Σ, m) a σ-finite measure space. Let there be defined on S a strongly measurable function T_s with values in the Banach space $\mathcal{B}(\mathfrak{X})$ of bounded linear operators on $\mathfrak{X}$. Suppose that $\|T_s\| \leq 1$ for all $s \in S$. Let h be a measure-preserving transformation in (S, Σ, m). Then for each $X \in L^1(S, m)$ there is an $\bar{X} \in L^1(S, m)$ such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} T_s T_{h(s)} \cdots T_{h^{i-1}(s)} X(h^i(s)) = \bar{X}(s)$$

strongly in $\mathfrak{X}$, a.e. in S, and

$$\bar{X}(s) = T_s(\bar{X}(h(s)))$$

a.e. in S. Moreover, if $m(S) < \infty$, then $\bar{X}$ is also the limit in the mean of order 1.

A Pointwise Ergodic Theorem for MRIS (I)

Assumption (STAT) One of the two following conditions is satisfied:

- π is stationary: $\pi P = \pi$.
- There exists a faithful stationary measure: $\pi_+ P = \pi_+ > 0$.

Theorem 1

Under Assumption **(STAT)**, for any $X \in \mathfrak{A}$, the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathcal{L}^*_{\omega_1} \cdots \mathcal{L}^*_{\omega_n} X(\omega_{n+1}) = \bar{X}(\boldsymbol{\omega})$$

exists $\mathbb{P}$-almost surely and in $L^1(\Omega^{\mathbb{N}}, \mathbb{P}; \mathfrak{A})$. The limiting function is such that

$$\mathcal{L}^*_{\omega} \bar{X}(\sigma(\boldsymbol{\omega})) = \bar{X}(\boldsymbol{\omega}).$$

Moreover, the extended observable $\bar{\bar{X}} \in \mathfrak{A}$ defined by

$$\bar{\bar{X}}(\omega) = \mathbb{E}[\overline{X}(\boldsymbol{\omega}) | \omega_1 = \omega],$$

satisfies

$$\mathbb{L}^* \bar{\bar{X}} = \bar{\bar{X}}.$$

A Pointwise Ergodic Theorem for MRIS (II)

Corollary 2

If $\mathbb{L}$ is **irreducible**, its unique ESS R_+ is faithful and for any $X \in \mathfrak{A}$ one has

$$\bar{X}(\boldsymbol{\omega}) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathcal{L}^*_{\omega_1} \cdots \mathcal{L}^*_{\omega_n} X(\omega_{n+1}) = \langle R_+, X \rangle \mathbb{1}$$

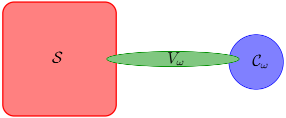
for $\mathbb{P}$-almost every $\omega \in \Omega$ and in $L^1(\Omega^{\mathbb{N}}, \mathbb{P}; \mathfrak{A})$. In particular, for any initial state $(\rho_\omega)_{\omega\in\Omega}$ and any $X \in \mathfrak{A}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \langle \rho_n(\boldsymbol{\omega}), X(\omega_{n+1}) \rangle = \langle R_+, X \rangle$$

holds $\mathbb{P}$-almost surely and in $L^1(\Omega^{\mathbb{N}}, \mathbb{P})$. Moreover, if $\mathbb{L}$ is **primitive**, then

$$\lim_{n\to\infty} \mathbb{E}_+[\langle \rho_n(\boldsymbol{\omega}), X(\omega_{n+1}) \rangle] = \langle R_+, X \rangle.$$

From now on we assume more structure:



$$\mathcal{L}_\omega \rho = \mathrm{tr}_{\mathcal{H}_{\mathcal{C}_\omega}} \left(U_\omega (\rho \otimes \rho_{\mathcal{C}_\omega}) U_\omega^* \right)$$

with

$$U_\omega = \mathrm{e}^{-\mathrm{i}\tau_\omega (H_{\mathcal{S}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathcal{C}_\omega} + V_\omega)}$$

and

$$\rho_{\mathcal{C}_\omega} = \mathrm{e}^{-\beta_\omega (H_{\mathcal{C}_\omega} - F_\omega)}$$

The Heat Currents

- Energy dissipated during the $n+1^{\text{th}}$ interaction

$$\Delta Q_{n+1}(\boldsymbol{\omega}) = \mathrm{tr}\left(\rho_n(\boldsymbol{\omega}) \otimes \rho_{\mathcal{C}_{\omega_{n+1}}}(U^*_{\omega_{n+1}} H_{\mathcal{C}_{\omega_{n+1}}} U_{\omega_{n+1}} - H_{\mathcal{C}_{\omega_{n+1}}})\right) = -\langle \rho_n(\boldsymbol{\omega}), J(\omega_{n+1})\rangle$$

with
$$J(\omega) = \mathrm{tr}_{\mathcal{H}_{\mathcal{C}_\omega}}(U^*_\omega [U_\omega, H_{\mathcal{C}_\omega}](\mathbb{1} \otimes \rho_{\mathcal{C}_\omega})).$$

- Further setting
$$J_\nu : \Omega \ni \omega \mapsto \delta_{\nu\omega} J(\nu),$$

yields an extended observable $J_\nu \in \mathfrak{A}$ describing the energy transferred from reservoir $\mathcal{R}_\nu$ to the system $\mathcal{S}$ during a single interaction.

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- Time-averaged quantum mechanical expectation of the heat extracted from reservoir $\mathcal{R}_\nu$ during a single interaction

$$\bar{J}_\nu(\boldsymbol{\omega}) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \langle \rho_n(\boldsymbol{\omega}), J_\nu(\omega_{n+1})\rangle$$

- If $\mathbb{L}$ is irreducible, $\mathbb{P}$-a.s.

$$\bar{J}_\nu(\boldsymbol{\omega}) = \langle R_+, J_\nu \rangle$$

Entropy Balance

- After n^{th} interaction, the von Neumann entropy of $\mathcal{S}$ is

$$S[\rho_n(\boldsymbol{\omega})] = -\langle \rho_n(\boldsymbol{\omega}), \log \rho_n(\boldsymbol{\omega}) \rangle.$$

- During the $n+1^{\text{th}}$ interaction, this entropy decreases by

$$\Delta S_{n+1}(\boldsymbol{\omega}) = S[\rho_n(\boldsymbol{\omega})] - S[\rho_{n+1}(\boldsymbol{\omega})].$$

- Entropy balance

$$\Delta S_{n+1}(\boldsymbol{\omega}) + \mathrm{ep}_{n+1}(\boldsymbol{\omega}) = \beta_{\omega_{n+1}} \Delta Q_{n+1}(\boldsymbol{\omega}),$$

  where, by definition, $\mathrm{ep}_{n+1}(\boldsymbol{\omega})$ is the entropy production associated with the $n+1^{\text{th}}$ interaction.

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- Recall: relative entropy of states is defined as

$$\mathrm{Ent}(\mu|\rho) = \langle \mu, \log \mu - \log \rho \rangle \geq 0$$

  with equality iff $\mu = \rho$.

Entropy Balance

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with equality iff $\mu = \rho$.

Lemma 3 (2^{nd}-Law of Thermodynamics)

$$\mathrm{ep}_{n+1}(\omega) = \mathrm{Ent}\left(U_{\omega_{n+1}}(\rho_n(\omega) \otimes \rho_{\mathcal{C}_{\omega_{n+1}}}) U^*_{\omega_{n+1}} \,\middle|\, \rho_{n+1}(\omega) \otimes \rho_{\mathcal{C}_{\omega_{n+1}}} \right) \geq 0$$

implies Landauer's bound

$$\Delta Q_{n+1}(\omega) \geq \frac{\Delta S_{n+1}(\omega)}{\beta_{\omega_{n+1}}}.$$

Thermodynamic Equilibrium

Time average of the entropy balance relation

$$\frac{S[\rho_0(\boldsymbol{\omega})] - S[\rho_N(\boldsymbol{\omega})]}{N} + \frac{1}{N}\sum_{n=0}^{N-1} \mathrm{ep}_n(\boldsymbol{\omega}) = -\sum_{\nu\in\Omega} \beta_\nu \frac{1}{N}\sum_{n=0}^{N-1} \langle \rho_n(\boldsymbol{\omega}), J_\nu(\omega_{n+1})\rangle$$

yields, in the limit $N \to \infty$

$$\bar{\mathrm{ep}}(\boldsymbol{\omega}) = -\sum_{\nu\in\Omega} \beta_\nu \bar{J}_\nu(\boldsymbol{\omega})$$

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yields, in the limit $N \to \infty$, if $\mathbb{L}$ is irreducible,

$$\bar{\mathrm{ep}}(\boldsymbol{\omega}) = -\sum_{\nu\in\Omega} \beta_\nu \bar{J}_\nu(\boldsymbol{\omega}) = -\sum_{\nu\in\Omega} \beta_\nu \langle R_+, J_\nu\rangle$$

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Equilibrium means **no heat currents** $\Longrightarrow$ **no entropy production**

Theorem 3

If $\mathbb{L}$ is irreducible, then $\bar{\mathrm{ep}}_n(\boldsymbol{\omega}) = 0$ holds $\mathbb{P}$-a.s. iff the family of states $(\rho_{+\omega})_{\omega\in\Omega}$ satisfies

$$U_\omega(\rho_{+\nu} \otimes \rho_{\mathcal{C}_\omega})U_\omega^* = \rho_{+\omega} \otimes \rho_{\mathcal{C}_\omega}$$

for all $\nu, \omega \in \Omega$ such that $P_{\nu\omega} > 0$. In this case the entropy balance reads

$$S[\rho_{+\omega_{n+1}}] - S[\rho_{+\omega_n}] = \beta_{\omega_{n+1}} \langle \rho_{+\omega_n}, J(\omega_{n+1})\rangle = 0$$

$\mathbb{P}$-a.s. and in particular

$$\langle R_+, J_\nu\rangle = 0$$

so that **no heat currents** $\Longleftrightarrow$ **no entropy production**

Full Statistics, Fluctuation Relations and Linear Response

Full Statistics of Environmental Entropy

Repeated two-time measurement protocol: Entropy observable

$$S_\omega = -\log \rho_{\mathcal{C}_\omega} = \beta_\omega (H_{\mathcal{C}_\omega} - F_\omega)$$

is measured before and after each interaction with $\mathcal{C}_\omega$ with outcome

$$\xi = (\varsigma, \varsigma') \in \Sigma \times \Sigma, \qquad \Sigma = \bigcup_{\omega \in \Omega} \mathrm{sp}(S_\omega).$$

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Lüders–Schwinger–Wigner formula

$$\mathbb{Q}(\xi_1, \ldots \xi_n|\boldsymbol{\omega}) = \langle \mathcal{L}_{\omega_n,\xi_n} \cdots \mathcal{L}_{\omega_1,\xi_1}\rho_{\omega_0}, \mathbb{1}\rangle$$

with

$$\mathcal{L}_{\omega,\xi}\rho = \mathrm{e}^{-\varsigma}\mathrm{tr}_{\mathcal{H}_{\mathcal{C}_\omega}}\left((\mathbb{1}\otimes \mathbb{1}_{\{S_\omega=\varsigma'\}})U_\omega(\rho\otimes \mathbb{1}_{\{S_\omega=\varsigma\}})U_\omega^*)\right)$$

gives the joint probability law of $\xi_1, \ldots, \xi_n$ after n interactions, conditioned on $\boldsymbol{\omega}$.

Full Statistics of Environmental Entropy

Repeated two-time measurement protocol: Entropy observable

$$S_\omega = -\log \rho_{\mathcal{C}_\omega} = \beta_\omega (H_{\mathcal{C}_\omega} - F_\omega)$$

is measured before and after each interaction with $\mathcal{C}_\omega$ with outcome

$$\xi = (\varsigma, \varsigma') \in \Sigma \times \Sigma, \qquad \Sigma = \bigcup_{\omega \in \Omega} \mathrm{sp}(S_\omega).$$

Lüders–Schwinger–Wigner formula

$$\mathbb{Q}(\xi_1, \ldots \xi_n | \boldsymbol{\omega}) = \langle \mathcal{L}_{\omega_n, \xi_n} \cdots \mathcal{L}_{\omega_1, \xi_1} \rho_{\omega_0}, \mathbb{1} \rangle$$

with

$$\mathcal{L}_{\omega,\xi}\rho = \mathrm{e}^{-\varsigma} \mathrm{tr}_{\mathcal{H}_{\mathcal{C}_\omega}} \left((\mathbb{1} \otimes \mathbb{1}_{\{S_\omega = \varsigma'\}}) U_\omega (\rho \otimes \mathbb{1}_{\{S_\omega = \varsigma\}}) U_\omega^* \right)$$

gives the joint probability law of $\xi_1, \ldots, \xi_n$ after n interactions, conditioned on $\boldsymbol{\omega}$.

$\mathbb{Q}$ extends to a probability on $(\Sigma \times \Sigma)^{\mathbb{N}}$, so we can make the following

Definition

The Full Statistics of Entropy is the probability measure

$$\widetilde{\mathbb{P}}(\mathrm{d}\boldsymbol{\xi}\mathrm{d}\boldsymbol{\omega}) = \mathbb{Q}(\mathrm{d}\boldsymbol{\xi}|\boldsymbol{\omega})\mathbb{P}(\mathrm{d}\boldsymbol{\omega})$$

on $(\Sigma \times \Sigma \times \Omega)^{\mathbb{N}}$.

Full Statistics of Environmental Entropy

Set $\delta\xi = \varsigma' - \varsigma$ for $\xi = (\varsigma, \varsigma')$

Full Statistics of Environmental Entropy

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The total increase of the entropy of the reservoirs after N interactions is

$$\boldsymbol{\mathcal{I}}_N = (\mathcal{I}_{N,\nu})_{\nu\in\Omega}, \qquad \mathcal{I}_{N,\nu} = \sum_{n=1}^{N} 1_{\{\omega_n=\nu\}}\delta\xi_n.$$

Full Statistics of Environmental Entropy

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$$\boldsymbol{\mathcal{I}}_N = (\mathcal{I}_{N,\nu})_{\nu\in\Omega}, \qquad \mathcal{I}_{N,\nu} = \sum_{n=1}^{N} 1_{\{\omega_n=\nu\}}\delta\xi_n.$$

$$\mathcal{L}_\omega^{[\alpha]} := \sum_{\xi\in\Sigma\times\Sigma} \mathrm{e}^{-\alpha\delta\xi}\mathcal{L}_{\omega,\xi}$$

$$(\mathbb{L}^{[\boldsymbol{\alpha}]}R)(\omega) := \sum_{\nu\in\Omega} P_{\nu\omega}\mathcal{L}_\nu^{[\alpha_\nu]}R(\nu) \qquad \boldsymbol{\alpha} = (\alpha_\omega)_{\omega\in\Omega} \in \mathbb{R}^\Omega$$

$$\ell(\boldsymbol{\alpha}) = \max\{|\lambda| \,|\, \lambda \in \mathrm{sp}(\mathbb{L}^{[\boldsymbol{\alpha}]})\}$$

Theorem 4

- Under Assumption **(STAT)**, one has

$$\lim_{N\to\infty} \widetilde{\mathbb{E}}\left[\frac{\mathcal{I}_{N,\nu}}{N}\right] = -\beta_\nu \mathbb{E}[\langle \rho_{\omega_0}, \bar{J}_\nu(\boldsymbol{\omega})\rangle].$$

- If $\mathbb{L}$ is irreducible, then the weak law of large numbers holds, i.e., the limit

$$\lim_{N\to\infty} \frac{\mathcal{I}_{N,\nu}}{N} = -\beta_\nu \langle R_+, J_\nu\rangle,$$

  exists in probability.
- If $\mathbb{L}$ is irreducible, then the central limit theorem holds, i.e., as $N \to \infty$

$$\frac{1}{\sqrt{N}}\left(\boldsymbol{\mathcal{I}}_N - \widetilde{\mathbb{E}}\,[\boldsymbol{\mathcal{I}}_N]\right)$$

  converges in law towards a centered Gaussian vector with covariance matrix

$$C_{\omega\nu} = \ell_{\omega\nu} - \ell_\omega \ell_\nu,$$

  where

$$\ell_\omega = (\partial_{\alpha_\omega}\ell)(0), \quad \ell_{\omega\nu} = (\partial_{\alpha_\nu}\partial_{\alpha_\omega}\ell)(0).$$

. . . and Large Deviation Principle

Theorem 4 (cont'd)

- If $\mathbb{L}$ is primitive, then the limit

$$e(\boldsymbol{\alpha}) = \lim_{N\to\infty} \frac{1}{N} \log \widetilde{\mathbb{E}}[\mathrm{e}^{-\boldsymbol{\alpha}\cdot\boldsymbol{\mathcal{I}}_N}]$$

  exists, defines a real analytic function. Moreover, for all $\boldsymbol{\alpha} \in \mathbb{R}^\Omega$,

$$e(\boldsymbol{\alpha}) = \log \ell(\boldsymbol{\alpha}).$$

- If $\mathbb{L}$ is primitive, then the sequence of random vectors $(\boldsymbol{\mathcal{I}}_N)_{N\in\mathbb{N}}$ satisfies a large deviation principle: for any Borel set $G \subset \mathbb{R}^\Omega$,

$$\begin{aligned} -\inf_{\varsigma\in\mathring{G}} I(\varsigma) &\le \liminf_{N\to\infty} \frac{1}{N} \log \widetilde{\mathbb{P}}\left(\frac{\boldsymbol{\mathcal{I}}_N}{N} \in G\right) \\ &\le \limsup_{N\to\infty} \frac{1}{N} \log \widetilde{\mathbb{P}}\left(\frac{\boldsymbol{\mathcal{I}}_N}{N} \in G\right) \le -\inf_{\varsigma\in\bar{G}} I(\varsigma), \end{aligned}$$

  where $\mathring{G}/\bar{G}$ denote the interior/closure of G and the good rate function $\varsigma \mapsto I(\varsigma)$ is given by the Legendre-Fenchel transform of the function $\boldsymbol{\alpha} \mapsto e(-\boldsymbol{\alpha})$,

$$I(\varsigma) := \sup_{\boldsymbol{\alpha}\in\mathbb{R}^\Omega} \left(\boldsymbol{\alpha}\cdot\varsigma - e(-\boldsymbol{\alpha})\right).$$

Fluctuation Relations

A strong form of Fluctuation Relations [Gallavotti-Cohen (1995)] holds under

Assumption (TRI) The two following conditions are satisfied:

- The driving Markov chain is reversible, i.e., satisfies the detailed balance condition: for all $\omega, \nu \in \Omega$,

$$\pi_\omega P_{\omega\nu} = \pi_\nu P_{\nu\omega}$$

- There are anti-unitary involutions θ and θ_ω acting on $\mathcal{H}_\mathcal{S}$ and $\mathcal{H}_{\mathcal{C}_\omega}$, such that

$$\theta_\omega H_{\mathcal{C}_\omega} = H_{\mathcal{C}_\omega}\theta_\omega, \qquad (\theta \otimes \theta_\omega)U_\omega = U_\omega^*(\theta \otimes \theta_\omega)$$

for all $\omega \in \Omega$.

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$$\theta_\omega H_{\mathcal{C}_\omega} = H_{\mathcal{C}_\omega}\theta_\omega, \qquad (\theta \otimes \theta_\omega)U_\omega = U_\omega^*(\theta \otimes \theta_\omega)$$

for all $\omega \in \Omega$.

Theorem 5

If $\mathbb{L}$ is primitive and Assumption **(TRI)** is satisfied, then the rate function governing the large deviations of the entropy full statistics satisfies the Fluctuation Relation

$$I(-\varsigma) - I(\varsigma) = \sum_{\omega\in\Omega} \varsigma_\omega, \qquad (\text{ for all } \varsigma \in \mathbb{R}^\Omega),$$

which is associated with the Gallavotti-Cohen symmetry ($\mathbf{1} = (1, 1, \ldots .1)$)

$$e(\mathbf{1} - \boldsymbol{\alpha}) = e(\boldsymbol{\alpha}), \qquad (\text{ for all } \boldsymbol{\omega} \in \mathbb{R}^\Omega).$$

Fluctuation Relations

The total entropy dumped in the environment after N interaction is

$$\sigma_N = \sum_{\omega\in\Omega} \varsigma_{N,\omega} = \mathcal{I}_N \cdot \mathbf{1}$$

Corollary 6

Under the Hypotheses of Theorem 5, the large deviation estimate

$$-\inf_{s\in\mathring{S}} \bar{I}(s) \le \liminf_{N\to\infty} \frac{1}{N}\log\widetilde{\mathbb{P}}\left(\frac{\sigma_N}{N}\in S\right) \le \limsup_{N\to\infty}\frac{1}{N}\log\widetilde{\mathbb{P}}\left(\frac{\sigma_N}{N}\in S\right) \le -\inf_{s\in\bar{S}} \bar{I}(s)$$

holds with rate

$$\bar{I}(s) = \inf\{I(\varsigma)\,|\,\varsigma\cdot\mathbf{1} = s\}$$

satisfying the Fluctuation Relation

$$\bar{I}(-s) - \bar{I}(s) = s$$

The last relation can be loosely formulated as

$$\frac{\widetilde{\mathbb{P}}(\sigma_N = -Ns)}{\widetilde{\mathbb{P}}(\sigma_N = Ns)} \simeq \mathrm{e}^{-Ns}$$

showing that negative values of entropy production are exponentially suppressed, a strong form of the 2nd Law of Thermodynamics.

Linear Response

Fluctuation relations as far from equilibrium extensions of fluctuation–dissipation relations (Green–Kubo, Onsager reciprocity) [Gallavotti (1996)].

Is there a notion of equilibrium for MRIS ?

Assumption (EQU) The three following conditions are satisfied:

- All reservoirs are at the same temperature: $\beta_\nu = \bar{\beta}$.
- $\mathbb{L}$ is irreducible with unique ESS R_+.
- Entropy production vanishes: $\sum_{\nu\in\Omega} \bar{\beta}\langle R_+, J_\nu\rangle = 0$.

Linear Response

Under Assumption **(EQU)**, let us perturb the reservoirs states

$$\rho_{\mathcal{C}_\omega,\zeta} = \mathrm{e}^{-(\bar\beta-\zeta_\omega)(H_{\mathcal{C}_\omega}-F_{\omega,\zeta})}, \qquad \zeta = (\zeta_\omega)_{\omega\in\Omega} \in \mathbb{R}^\Omega,$$

and denote by the subscript ζ the correspondingly perturbed quantities. The following is a strong form of the 1st Law of Thermodynamics (energy conservation).

Theorem 6

Under Assumption **(EQU)**, $\mathbb{L}_\zeta$ is irreducible. It is primitive whenever $\mathbb{L}$ is.

- For $\mathbb{P}_\zeta$ a.e. $\boldsymbol{\omega} \in \Omega^{\mathbb{N}}$

$$\sum_{\nu\in\Omega} \bar{J}_{\nu\zeta}(\boldsymbol{\omega}) = \langle R_{+\zeta}, J_{\nu\zeta}\rangle = 0.$$

- Under the law $\widetilde{\mathbb{P}}_\zeta$ the limit

$$\lim_{N\to\infty} \frac{1}{N}\sum_{\nu\in\Omega}(\bar\beta-\zeta_\nu)^{-1}\mathcal{I}_{N,\nu\zeta} = 0$$

  holds in probability.
- The Gaussian measure obtained in Theorem 4 as the limiting law of

$$\frac{1}{\sqrt{N}}\left(\boldsymbol{\mathcal{I}}_{N\zeta} - \mathbb{E}_\zeta[\boldsymbol{\mathcal{I}}_{N\zeta}]\right)$$

  as $N\to\infty$ is supported by the hyperplane $\mathfrak{z}_\zeta = \{\varsigma \mid \sum_{\nu\in\Omega}(\bar\beta-\zeta_\nu)^{-1}\varsigma_\nu = 0\}$.

Linear Response

Theorem 6 Cont'd

Suppose in addition that $\mathbb{L}$ is primitive.

- The cumulant generating function has a translation symmetry: for all $\boldsymbol{\alpha} \in \mathbb{R}^{\Omega}$ and $\gamma \in \mathbb{R}$
$$e_{\zeta}(\boldsymbol{\alpha} + \gamma\boldsymbol{\beta}^{-1}) = e_{\zeta}(\boldsymbol{\alpha}), \qquad \boldsymbol{\beta}^{-1} = ((\bar{\beta} - \zeta_{\omega})^{-1})_{\omega\in\Omega}.$$
- The rate function of the large deviation principle of the full statistics of entropy satisfies
$$I_{\zeta}(\varsigma) = +\infty$$
for $\varsigma \notin \mathfrak{z}_{\zeta}$.

Linear Response

Set

$$S_{+\omega} = -\log \rho_{+\omega} = -\log \frac{\mathcal{L}_\omega R_+(\omega)}{\mathrm{tr}\, R_+(\omega)},$$

$$\widehat{J}_\nu(\omega) = \delta_{\omega\nu} \mathrm{tr}_{\mathcal{C}_\omega} \left((U_\omega H_{\mathcal{C}_\omega} U_\omega^* - H_{\mathcal{C}_\omega})(\mathbb{1} \otimes \rho_{\mathcal{C}_\omega}) \right),$$

$$\widehat{\mathcal{L}}_\omega \rho = \mathrm{tr}_{\mathcal{C}_\omega} \left(U_\omega^* (\rho \otimes \rho_{\mathcal{C}_\omega}) U_\omega \right)$$

Linear Response

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$$S_{+\omega} = -\log \rho_{+\omega} = -\log \frac{\mathcal{L}_\omega R_+(\omega)}{\mathrm{tr}\, R_+(\omega)},$$

$$\widehat{J}_\nu(\omega) = \delta_{\omega\nu} \mathrm{tr}_{\mathcal{C}_\omega} \left((U_\omega H_{\mathcal{C}_\omega} U_\omega^* - H_{\mathcal{C}_\omega})(\mathbb{1} \otimes \rho_{\mathcal{C}_\omega}) \right),$$

$$\widehat{\mathcal{L}}_\omega \rho = \mathrm{tr}_{\mathcal{C}_\omega} \left(U_\omega^* (\rho \otimes \rho_{\mathcal{C}_\omega}) U_\omega \right)$$

Remark. If **(TRI)** holds and $\Theta : X \mapsto \theta X \theta$, then

$$\widehat{J}_\nu = -\Theta J_\nu, \qquad \widehat{\mathcal{L}}_\omega = \Theta \mathcal{L}_\omega \Theta$$

Linear Response

Set

$$S_{+\omega} = -\log \rho_{+\omega} = -\log \frac{\mathcal{L}_\omega R_+(\omega)}{\mathrm{tr}\, R_+(\omega)},$$

$$\widehat{J}_\nu(\omega) = \delta_{\omega\nu} \mathrm{tr}_{\mathcal{C}_\omega} \left((U_\omega H_{\mathcal{C}_\omega} U_\omega^* - H_{\mathcal{C}_\omega})(\mathbb{1} \otimes \rho_{\mathcal{C}_\omega}) \right),$$

Theorem 7

- If, in addition to the previous Assumptions **(TRI)** also holds, then the kinetic coefficients

$$L_{\omega\nu} = \partial_{\zeta_\nu} \langle R_{+\zeta}, J_{\omega\zeta} \rangle \big|_{\zeta = \mathbf{0}}$$

  are given by the Green-Kubo formula

$$L\omega\nu = \frac{1}{2} \sum_{n\in\mathbb{N}} \mathbb{E}_+[\langle \rho_{+\omega_0}, \widehat{J}_\nu(\omega_0) \mathcal{L}_{\omega_1}^* \cdots \mathcal{L}_{\omega_n}^* J_\omega(\omega_{n+1}) \rangle + \langle \omega \leftrightarrow \nu \rangle]$$

$$+\delta_{\omega\nu} \mathbb{E}_+[\langle \rho_{+\omega_0}, \widehat{\mathcal{L}}_{\omega_1}^*(S_{+\omega_1}^2) + \widehat{\mathcal{L}}_{\omega_1}^*(S_{+\omega_1}) S_{+\omega_0} + S_{+\omega_0} \widehat{\mathcal{L}}_{\omega_1}^*(S_{+\omega_1}) + S_{+\omega_0}^2 \rangle 1_{\omega_1 = \omega}]$$

  all the quantities on the right-hand side being evaluated at $\zeta = \mathbf{0}$.

- The Onsager reciprocity relation $L_{\omega\nu} = L_{\nu\omega}$ holds. Moreover, the kinetic coefficients are related to the covariance of the CLT in Theorem 4 by the fluctuation–dissipation relation

$$L_{\omega\nu} = \frac{1}{2\bar{\beta}^2} C_{\omega\nu}.$$

Outlook

Outlook

- Achievements
  - Extension of the main results of [Bougron–Bruneau 2020] to random RIS driven by Markov chains:
  - A pointwise ergodic theorem for abstract MRIS
  - Thermodynamics of MRIS under irreducibility: 1$^{\mathrm{st}}$ and 2$^{\mathrm{nd}}$ law, limit theorems for the full statistics of entropy/heat.
  - Characterization of the vanishing of entropy production.
  - Detailed fluctuation theorem (à la Gallavotti–Cohen), including linear response, under primitivity.
- Open Questions
  - Derive a fluctuation theorem under less stringent assumptions
  - Investigate the possibility of occurrence of phase transitions (non-analyticity of the cumulant generating function $e(\boldsymbol{\alpha})$).
  - Find non-trivial examples of MRIS with vanishing entropy production, i.e., examples with $\rho_{+\omega} \neq \rho_{+\nu}$ for distinct ω, ν (or prove that they do not exist!).

Thank You!