Thermodynamics of Repeated Quantum Measurements

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Stochastic and Analytic Methods in Mathematical Physics
Yerevan, Armenia, September 2-7, 2019

Irreversibility vs Measurements

- 1927: Heisenberg *"reduction of the wave function"*
- 1927: Eddington *"time's arrow"*
- 1932: von Neumann *"quantum arrow of time"*
- 1937: Landau-Lifschitz *"quantum vs thermodynamic arrow of time"*
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- 1963; Wigner *"causal vs statistical evolution"*
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- 1991: Zurek *"environment* $\Rightarrow$ *decoherence"*
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- More recently: *"statistical mechanics of repeated measurements"* Kümmerer-Maassen'04, Barchielli-Gregoratti'09, Bauer-Benoist-Bernard'11, Benoist-Pellegrini'14, Ballesteros-Fraas-Fröhlich-Schubnel'16,...

Other Motivations

- Non-demolition measurements (Braginsky, ..., Haroche Nobel Prize 2012)
- Finitely correlated states (Fannes, Nachtergaele, Werner 1992)
- Novel class of dynamical systems with surprising properties
- Subadditive thermodynamic formalism (Falconer, Barreira, Feng,...)
- Non-Gibbsian measures in SM (Dobrushin, Shlosman, van Enter, ...)

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- **Fluctuation relations for entropy production** (not in this talk)

Framework

Quantum Measurements & Quantum Instruments

- Quantum system $\longrightarrow \mathcal{H}$, $\dim\mathcal{H} < \infty$
- Observables $\longrightarrow \mathcal{B}(\mathcal{H})$, inner product $\langle X, Y\rangle = \mathrm{tr}(X^* Y)$
- Possible outcomes of a single measurement $\longrightarrow \mathcal{A} = \{a_1, \ldots, a_\ell\}$
- Quantum instrument $\{\Phi_a\}_{a\in\mathcal{A}}$
  - CP maps $\Phi_a : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$
  - $\Phi = \sum_{a\in\mathcal{A}} \Phi_a$ satisfies $\Phi(I) = I$
  - Duality $\langle \Phi_a^*(X), Y\rangle = \langle X, \Phi_a(Y)\rangle$
- Initial state ρ

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Operation Rules

- Probability of outcome $a \in \mathcal{A} \longrightarrow \mathrm{tr}(\Phi_a^*(\rho))$
- State after completion of measurement, conditioned on outcome $a \longrightarrow \dfrac{\Phi_a^*(\rho)}{\mathrm{tr}(\Phi_a^*(\rho))}$

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Remark

Any quantum instrument has a (non-unique) Kraus representation

$$\Phi_a(X) = \sum_{b\in\mathcal{B}_a} V_{a,b} X V_{a,b}^*, \qquad \Phi_a^*(X) = \sum_{b\in\mathcal{B}_a} V_{a,b}^* X V_{a,b}$$

Whenever such a representation exists with $|\mathcal{B}_a| = 1$ for all $a \in \mathcal{A}$, the instrument is called perfect.

Example 1

von Neumann Instrument $==$ Projective Measurement

- Observable: $A \in \mathcal{B}(\mathcal{H})$
- Spectral decomposition: $A = \sum_{a \in \mathcal{A}} aP_a$, with $\mathcal{A} = \mathrm{sp}(A)$
- Propagator from initial to measurement time: unitary $U \in \mathcal{B}(\mathcal{H})$
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- Probability of outcome $a \in \mathrm{sp}(A)$: $\mathrm{tr}(P_a U \rho U^*)$
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Remark

von Neumann instruments are perfect.

Example 2

General Instrument $==$ Ancila Measurement

- Probe $==$ quantum system with finite dimensional Hilbert space $\mathcal{H}_\mathrm{p}$
- Initial state of the probe: ρ_p
- Observable: $A \in \mathcal{B}(\mathcal{H}_\mathrm{p})$
- Spectral decomposition: $A = \sum_{a \in \mathcal{A}} a P_a$, with $\mathcal{A} = \mathrm{sp}(A)$
- Propagator from initial to measurement time: unitary $U \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}_\mathrm{p})$
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Any quantum instrument on $\mathcal{B}(\mathcal{H})$ can be realized with an appropriate ancila.

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- The probability of a finite "quantum trajectory" $\omega = (\omega_1, \ldots, \omega_T) \in \mathcal{A}^T$ is

$$\mathbb{P}_T(\omega) = \mathrm{tr}(\Phi^*_{\omega_T} \circ \cdots \circ \Phi^*_{\omega_2} \circ \Phi^*_{\omega_1}(\rho))$$

  (Lüders-Schwinger-Wigner formula)

Induced Classical Dynamical System

$$\mathbb{P}_T(\omega_1 \cdots \omega_T) = \langle I, \Phi^*_{\omega_T} \cdots \Phi^*_{\omega_1} \rho \rangle = \langle \Phi_{\omega_1} \cdots \Phi_{\omega_T} I, \rho \rangle$$

Recall $\Phi(I) = I$

- It implies

$$\sum_{\nu \in \mathcal{A}} \mathbb{P}_{T+1}(\omega_1 \cdots \omega_T \nu) = \sum_{\nu \in \mathcal{A}} \langle \Phi_\nu I, \Phi^*_{\omega_T} \cdots \Phi^*_{\omega_1} \rho \rangle = \mathbb{P}_T(\omega_1 \cdots \omega_T)$$

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- $\mathbb{P}$ is invariant under the left shift $\tau : \omega_1 \omega_2 \cdots \mapsto \omega_2 \omega_3 \cdots$ on Ω
- Dynamical system $(\Omega, \tau, \mathbb{P})$

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Assumption B

Φ is irreducible, i.e., there is no proper projection P such that $\Phi(P) \geq \lambda P$ for some $\lambda > 0$.

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Proposition

Under these assumptions the induced dynamical system $(\Omega, \tau, \mathbb{P})$ is ergodic

Entropies

- Basic quantities: the *"entropy functions"*

$$\Omega \ni \omega \mapsto S_T(\omega) = -\log \mathbb{P}_T(\omega_1 \cdots \omega_T)$$

and the Gibbs-Shannon entropies

$$\mathrm{Ent}(\mathbb{P}_T) = \mathbb{E}[S_T] = -\sum_{\omega \in \mathcal{A}^T} \mathbb{P}_T(\omega) \log \mathbb{P}_T(\omega)$$

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Results from ergodic theory

- Kolmogorov-Sinai: metric entropy of $\mathbb{P}$ w.r.t. the shift τ is

$$h_\tau(\mathbb{P}) = \lim_{T\to\infty} \frac{1}{T} \mathrm{Ent}(\mathbb{P}_T) \in [0, \log |\mathcal{A}|]$$

- Shannon-McMillan-Breiman:

$$\lim_{T\to\infty} \frac{1}{T} S_T(\omega) = h_\tau(\mathbb{P})$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$ and in $L^1(\Omega, \mathbb{P})$

Beyond the Shannon-McMillan-Breiman theorem

- **Large Deviations.** Quantify fluctuations around the SMB-limit

$$\frac{1}{T} S_T(\omega) \longrightarrow h_\tau(\mathbb{P})$$

by a LDP

$$\mathbb{P}_T\left[\frac{1}{T} S_T \sim s\right] \sim \mathrm{e}^{-T\, I(s)}$$

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- **Multifractal Analysis.** Fractal dimension of the level sets

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- **Statistical Mechanics.** Think of $\omega_1 \cdots \omega_T$ as a configuration of a spin chain. Look at $T \to \infty$ as a thermodynamic limit and develop the statistical mechanics of the infinite volume spin system: phase transitions, long range order ...

Pressure == Rényi Entropy

$$P_T(\beta) = \log \sum_{\omega \in \mathcal{A}^T} \mathbb{P}_T(\omega)^\beta = \log \sum_{\omega \in \mathcal{A}^T} \mathrm{e}^{-\beta S_T(\omega)}$$

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Theorem 1 [Thermodynamic formalism for $\beta > 0$]

1. For any $\mathbb{Q} \in \mathcal{P}_\tau(\Omega)$ the following limit exists

$$\varsigma(\mathbb{Q}) = \lim_{t \to \infty} \frac{1}{T} \int S_T(\omega) \mathrm{d}\mathbb{Q}(\omega)$$

2. For $\beta > 0$ one has

$$p(\beta) = \lim_{T \to \infty} \frac{1}{T} P_T(\beta) = \sup_{\mathbb{Q} \in \mathcal{P}_\tau(\Omega)} \left(h_\tau(\mathbb{Q}) - \beta\varsigma(\mathbb{Q})\right)$$

which defines a differentiable function.

3. For $\beta > 0$ there is a unique *equilibrium measure* $\mathbb{P}_\beta \in \mathcal{P}_\tau(\Omega)$ such that

$$p(\beta) = h_\tau(\mathbb{P}_\beta) - \beta\varsigma(\mathbb{P}_\beta)$$

Subadditive Thermodynamic Formalism

- **Upper decoupling:** Assumption (A) $\Longrightarrow$

$$\mathbb{P}_{T+T'}(\omega\omega') \le \rho_0^{-1}\mathbb{P}_T(\omega)\mathbb{P}_{T'}(\omega')$$

with $\rho_0 = \min \mathrm{sp}(\rho) \Longrightarrow$ super-additivity

$$S_{T+T'} \ge S_T + S_{T'} \circ \tau^T + \log \rho_0$$

which suffices to prove existence of limits (Fekete's Lemma).

- **Lower decoupling:** Assumption (B) $\Longrightarrow$ There is $C > 0$ and $t_l > 0$ such that for any finite words ω, ω' one can find a word ν of length $L \le t_l$ such that

$$\mathbb{P}_{T+L+T'}(\omega\nu\omega') \ge C\mathbb{P}_T(\omega)\mathbb{P}_{T'}(\omega')$$

This yields differentiability of the limit and uniqueness of equilibrium measure

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Theorem 1 $\Longrightarrow$ local LDP and Multifractal Formalism for $s \in]p'(0+), p'(+\infty)[$.

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- $\mathcal{H} = \mathbb{C}^2$, $\mathcal{A} = \{-, 0, +\}$, $\rho = I/2$
- Let R_θ be the rotation by θ, and $P_\pm$ the projections on the standard basis of $\mathbb{C}^2$

$$\Phi_0(X) = \frac{1}{2} R_\theta X R_\theta^*, \qquad \Phi_\pm(X) = \frac{1}{2} P_\pm X P_\pm$$

- For a.e. $\theta \in [0, 2\pi]$, $p(\beta)$ is finite for all $\beta \in \mathbb{R}$
- For a dense set of θ, $p(\beta) = +\infty$ for all $\beta < 0$

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- Pressure $\mathbb{R} \ni \beta \mapsto p(\beta) \longleftrightarrow$ Large Deviations (Gärtner-Ellis)

- Suppose there is a basis of $\mathcal{H}$ and a Kraus representation of the Φ_a such that all Kraus matrices have **algebraic** entries. Then $p(\beta) < \infty$ for all $\beta \in \mathbb{R}$

- $\mathcal{H} = \mathbb{C}^2$, $\mathcal{A} = \{-, 0, +\}$, $\rho = I/2$
- Let R_θ be the rotation by θ, and $P_\pm$ the projections on the standard basis of $\mathbb{C}^2$

$$\Phi_0(X) = \frac{1}{2} R_\theta X R_\theta^*, \qquad \Phi_\pm(X) = \frac{1}{2} P_\pm X P_\pm$$

- For a.e. $\theta \in [0, 2\pi]$, $p(\beta)$ is finite for all $\beta \in \mathbb{R}$
- For a dense set of θ, $p(\beta) = +\infty$ for all $\beta < 0$

In general, there is no thermodynamic formalism for $\beta < 0$

Main Results

Theorem II

1. **LDP for the entropy function.** For any Borel set Σ

$$-\inf_{s\in\dot{\Sigma}} I(s) \leq \liminf_{T\to\infty} \frac{1}{T}\log \mathbb{P}\left[\frac{S_T}{T}\in\Sigma\right] \leq \limsup_{T\to\infty} \frac{1}{T}\log \mathbb{P}\left[\frac{S_T}{T}\in\Sigma\right] \leq -\inf_{s\in\bar{\Sigma}} I(s)$$

   holds with rate function $I(s) = \sup_{\beta\in\mathbb{R}}(\beta s - p(-\beta))$

2. **Multifractal analysis of the entropy function.**

$$L_s \neq \emptyset \quad \Longrightarrow \quad \dim_H L_s = \frac{I(s)+s}{\log|\mathcal{A}|}$$

3. **Level II LDP.** The empirical measures

$$\mu_T^\omega = \frac{1}{T}\sum_{t=0}^{T-1}\delta_{\tau^t(\omega)}$$

   also satisfy the LDP with rate function

$$\mathbb{I}(\mathbb{Q}) = \begin{cases} \sup\limits_{v\in C(\Omega)}\left(\int v\,\mathrm{d}\mathbb{Q} - P(v)\right) & \text{if } \mathbb{Q}\in\mathcal{P}_\tau(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

   where $P(v) = \lim\limits_{T\to\infty}\frac{1}{T}\log\int \mathrm{e}^{\sum_{t=0}^{T-1} v\circ\tau^t}\mathrm{d}\mathbb{P}$

Example I: The Farey Instrument

$$\theta \in]0,2[$$

- $\mathcal{H} = \mathbb{C}^2$, $\mathcal{A} = \{0,1\}$
- $$\Phi_0(X) = \frac{1}{2+\theta}\begin{pmatrix} X_{11} + \theta X_{22} & 0 \\ 0 & \theta X_{22} \end{pmatrix} \qquad \Phi_1(X) = \frac{1}{2+\theta}\begin{pmatrix} X_{11} & 0 \\ 0 & (2-\theta)X_{11} + \theta X_{22} \end{pmatrix}$$
- $$\rho = \begin{pmatrix} 1 - \frac{\theta}{2} & 0 \\ 0 & \frac{\theta}{2} \end{pmatrix}$$

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- $$\rho = \begin{pmatrix} 1-\frac{\theta}{2} & 0 \\ 0 & \frac{\theta}{2}\end{pmatrix}$$
- $\mathbb{P}$ is a *matrix product state*

  $$\mathbb{P}_T(\omega_1\cdots\omega_T) = (2+\theta)^{-T}(1-\theta/2,\theta/2)M_{\omega_1}\cdots M_{\omega_T}\begin{pmatrix}1\\1\end{pmatrix}$$

  where

  $$M_0 = \begin{pmatrix} 1 & \theta \\ 0 & \theta\end{pmatrix} \qquad M_1 = \begin{pmatrix} 1 & 0 \\ 2-\theta & \theta\end{pmatrix}$$

Example I: The Farey Instrument

$\theta \neq 1$

- $\mathbb{R} \ni \beta \mapsto p(\beta)$ is real analytic
- $\mathbb{P} \sim$ equilibrium state of a spin system with exponentially decaying interactions
- No phase transition!

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Similar *number theoretic* spin chains have been extensively studied in 1990–2010 [Knauf,Kleban-Ozluk,...]

Example 2: The Erdös Instrument

- $\mathcal{H} = \mathbb{C}^2$, $\mathcal{A} = \{0, 1, 2\}$
- $\mathbb{P}$ is again a *matrix product state*

$$\mathbb{P}_T(\omega_1 \cdots \omega_T) = 5^{-T}(1/2, 1/2) M_{\omega_1} \cdots M_{\omega_T} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

where

$$M_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

- $\mathbb{P}$ is weak Gibbs with continuous potential
- The pressure $\mathbb{R} \ni \beta \mapsto p(\beta)$ is real analytic and strictly convex, except for a 1st order phase transition at $\beta_{\rm crit} \in [-3, -2]$

Example 3: The Keep-Switch Instrument

- A perfect Kraus instrument: $\mathcal{H} = \mathbb{C}^2$, $\mathcal{A} = \{-,+\}$, $\Phi_\pm(X) = V_\pm X V_\pm^*$

$$V_- = \begin{pmatrix} \sqrt{\cos\theta} & -\sin\theta/2 \\ -\sin\theta/2 & 0 \end{pmatrix}, \quad V_+ = \begin{pmatrix} -\sin\theta/2 & 0 \\ -\sqrt{\cos\theta} & -\sin\theta/2 \end{pmatrix}$$

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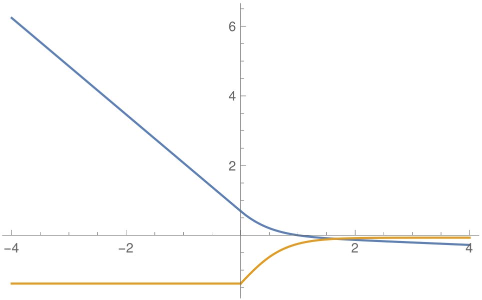

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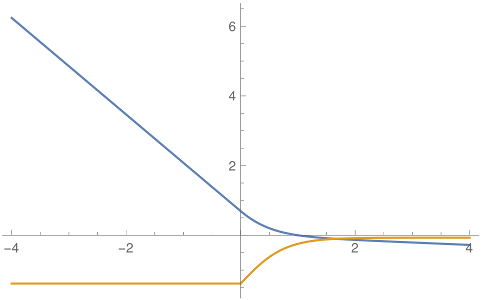

- Non Gaussian central limit theorem as $T \to \infty$

Perspectives

- Theory of 2 instruments: comparison/hypothesis testing (relative entropies), fluctuation theorems. . .
- Further develop the thermodynamic formalism for non-Gibbsian systems using results from the subadditive ergodic theory.
- Investigate the physical meaning of phase transition beyond the failure of CLT. Occurrence of anomalous scaling ?
- Special measurements, e.g., thermal probes.
- Continuous measurements/monitoring.
- Many instruments, parameter estimation (under development)
- . . .

Papers

- Benoist, Jakšić, Pautrat, P.: On entropy production of repeated quantum measurements I: General Theory. Commun. Math. Phys. 2017
- Cuneo, Jakšić, P., Shirikyan: Large deviations and fluctuation theorem for selectively decoupled measures on shift spaces. Rev. Math. Phys. 2019
- Benoist, Cuneo, Jakšić, P.: On entropy production of repeated quantum measurements II and III: Examples (soon on arXiv)
- ...

Thank you !